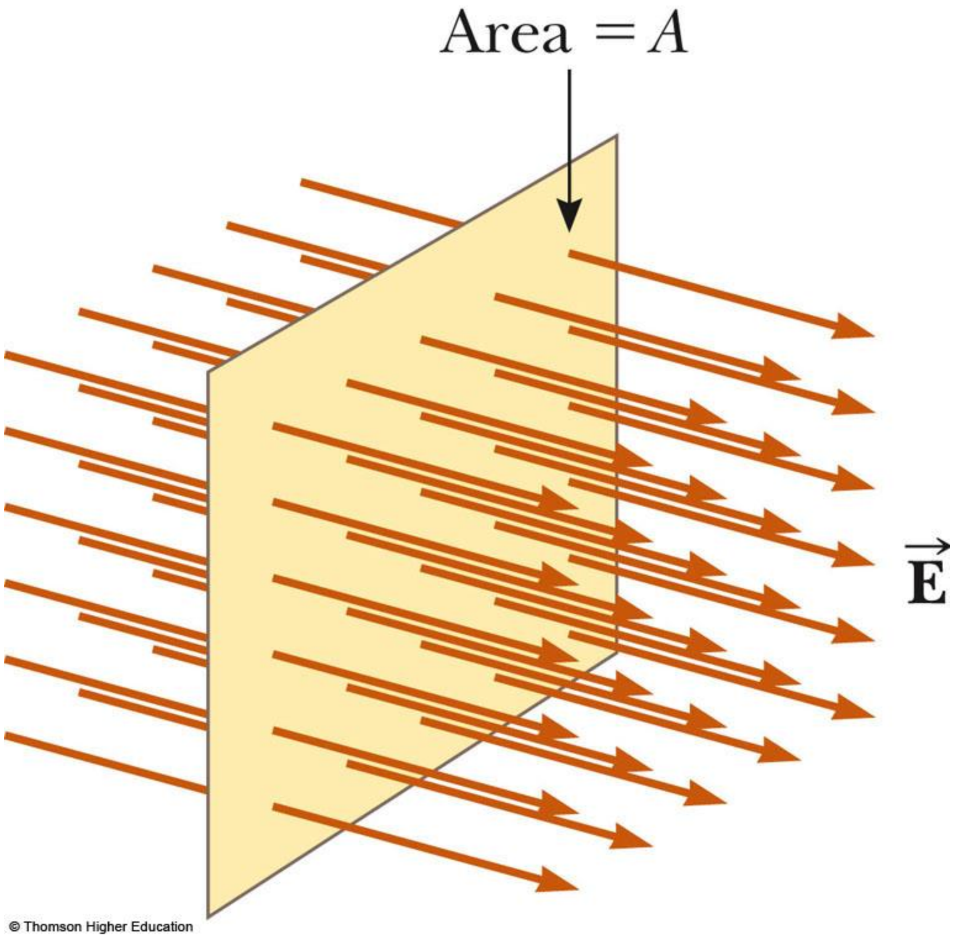
Gauss’s Law

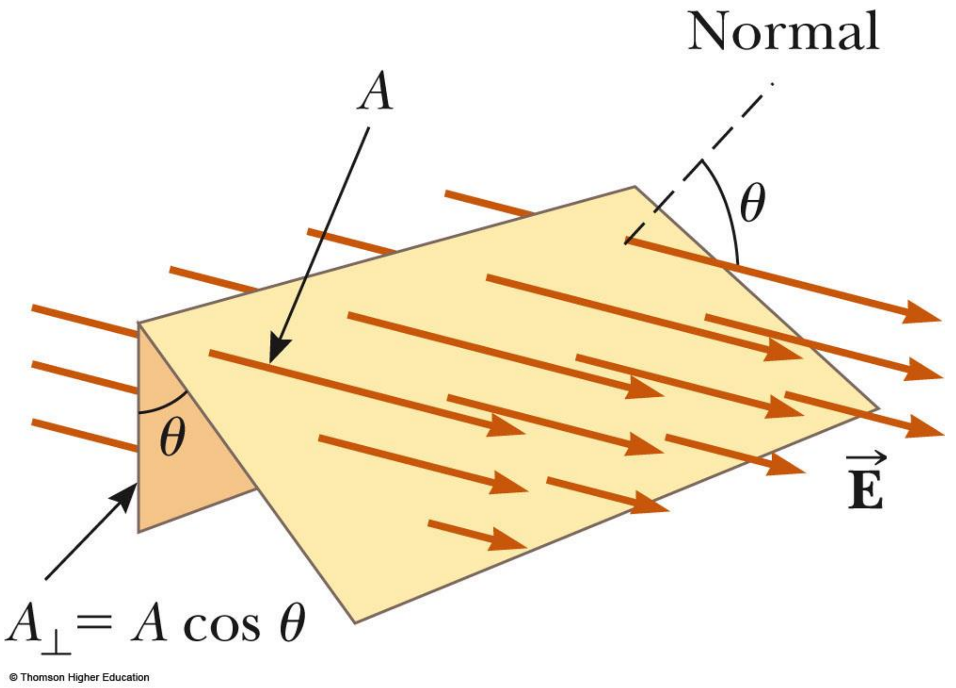
24.1 Electric Flux

- ***Electric flux*** is the product of the magnitude of the electric field and the surface area, *A*, perpendicular to the field
- $\Phi_E = EA$



Electric Flux, General Area

- The electric flux is proportional to the number of electric field lines penetrating some surface
- The field lines may make some angle θ with the perpendicular to the surface
- Then $\Phi_E = EA \cos \theta$



Electric Flux, Interpreting the Equation

- The flux is a maximum when the surface is perpendicular to the field
- The flux is zero when the surface is parallel to the field
- If the field varies over the surface, $\Phi = EA \cos \theta$ is valid for only a small element of the area

Electric Flux, General

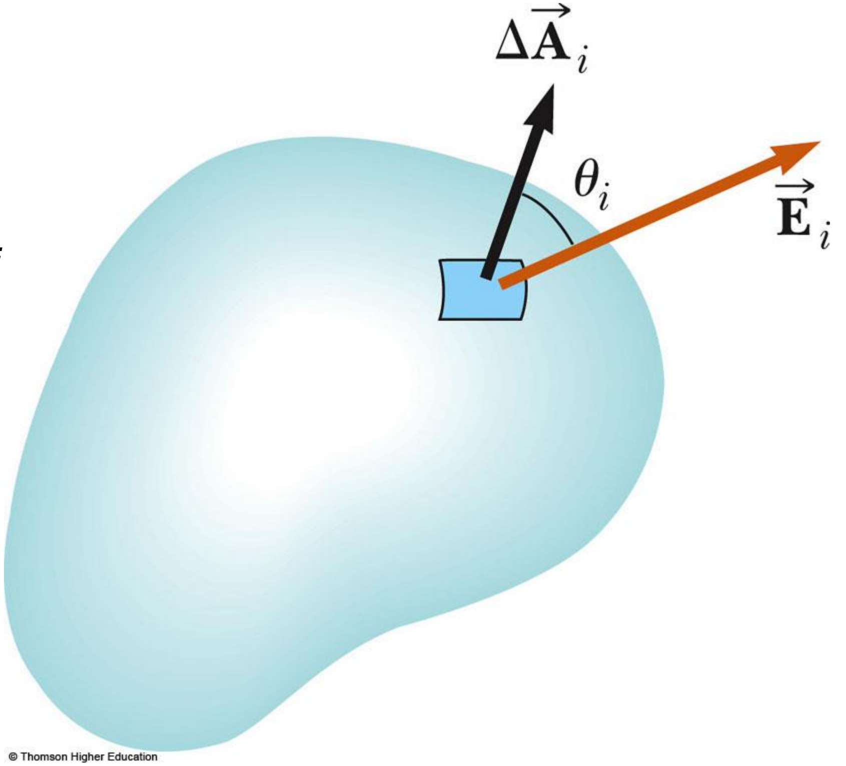
- In the more general case, look at a small area element

$$\Delta\Phi_E = E_i \Delta A_i \cos\theta_i = \vec{\mathbf{E}}_i \cdot \Delta\vec{\mathbf{A}}_i$$

- In general, this becomes

$$\Phi_E = \lim_{\Delta A_i \to 0} \sum E_i \cdot \Delta A_i$$

$$\Phi_E = \int_{\text{surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

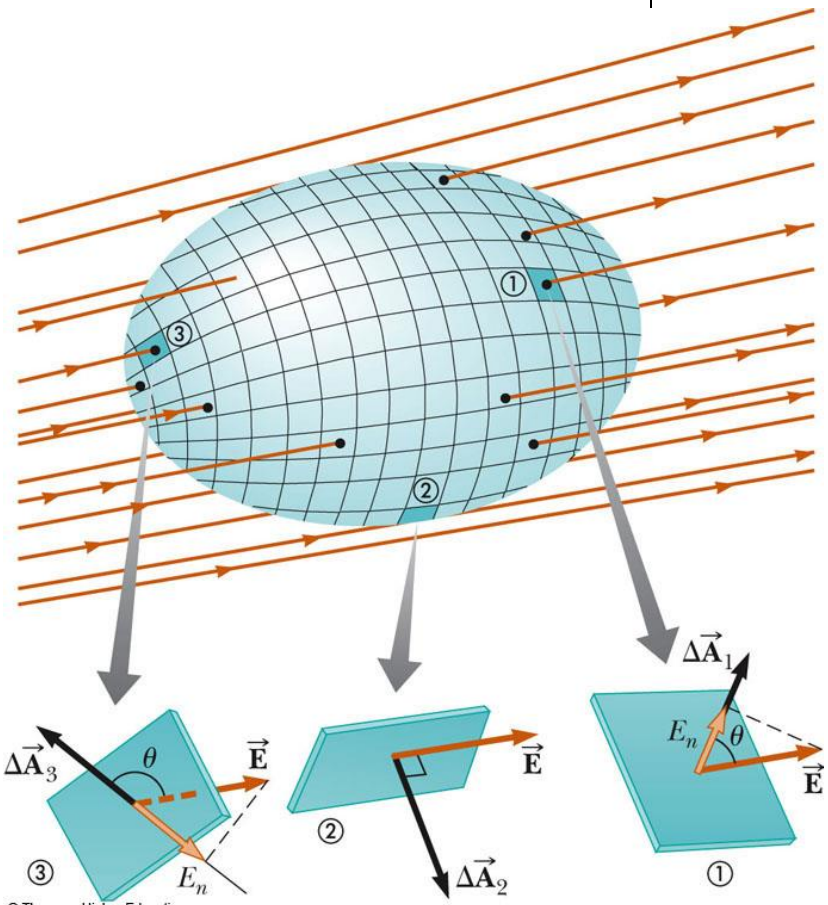


Electric Flux, final

- The surface integral means the integral must be evaluated over the surface in question
- In general, the value of the flux will depend both on the field pattern and on the surface
- The units of electric flux will be $N \cdot m^2/C^2$

Electric Flux, Closed Surface

- Assume a closed surface
- The vectors $\Delta\vec{\mathbf{A}}_i$ point in different directions
  - At each point, they are perpendicular to the surface
  - By convention, they point outward



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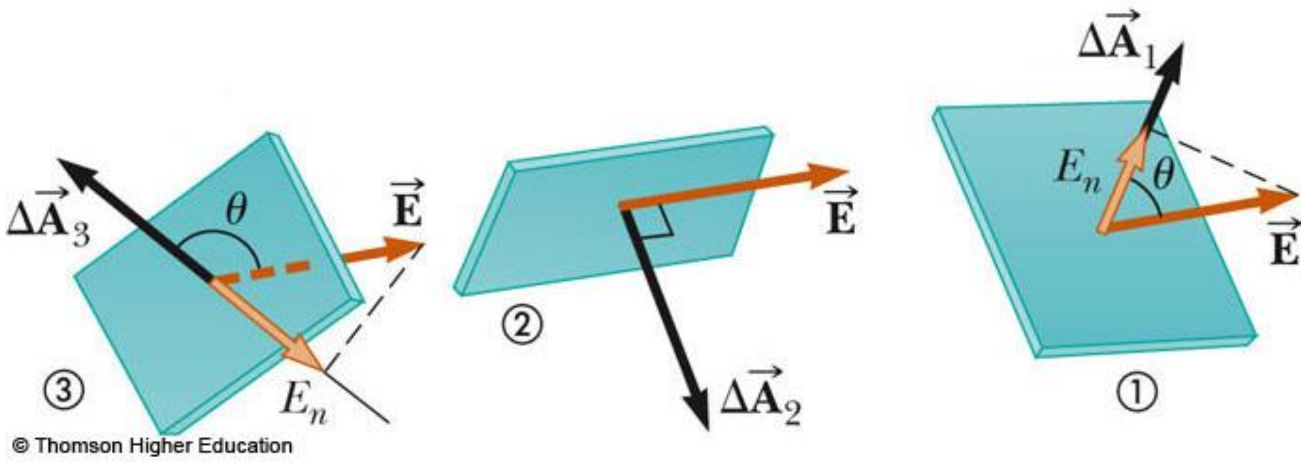
Flux Through Closed Surface, cont.



- At (1), the field lines are crossing the surface from the inside to the outside; $\theta < 90^o$, Φ is positive
- At (2), the field lines graze surface; $\theta = 90^o$, $\Phi = 0$
- At (3), the field lines are crossing the surface from the outside to the inside; $180^o > \theta > 90^o$, Φ is negative

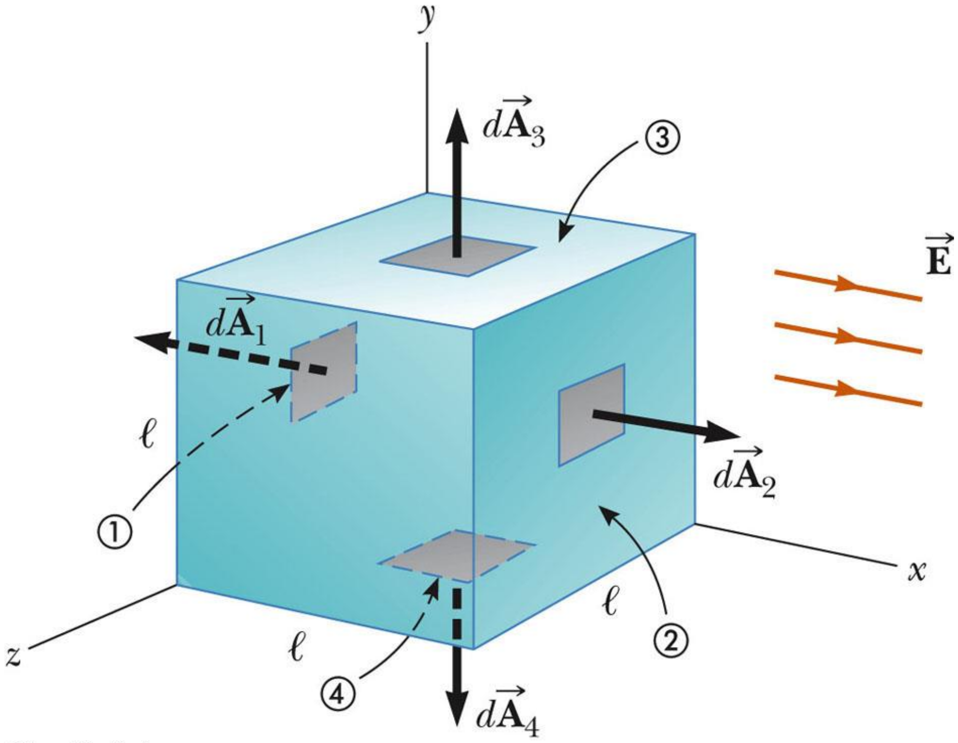
Flux Through Closed Surface, final

- The ***net*** flux through the surface is proportional to the net number of lines leaving the surface
  - This net number of lines is the number of lines leaving the surface minus the number entering the surface
- If E_n is the component of **E** perpendicular to the surface, then

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E_n dA$$

Flux Through a Cube, Example

- The field lines pass through two surfaces perpendicularly and are parallel to the other four surfaces
- For side 1, $E = -E\ell^2$
- For side 2, $E = E\ell^2$
- For the other sides, $E = 0$
- Therefore, $E_{total} = 0$



Karl Friedrich Gauss

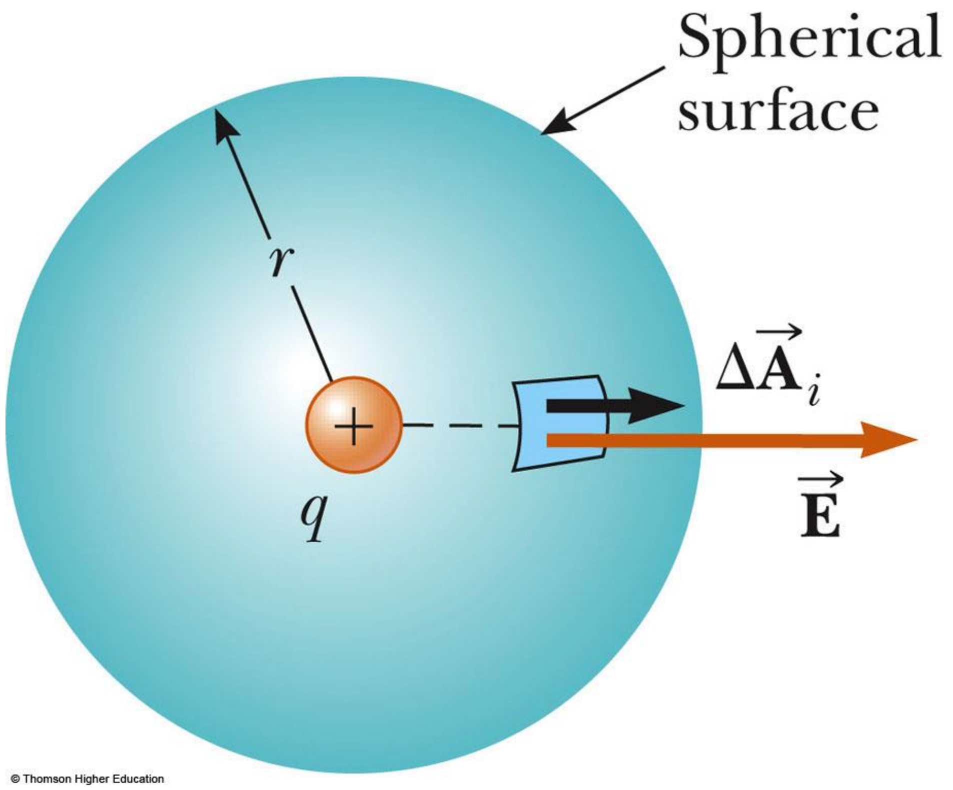
- 1777 – 1855
- Made contributions in
  - Electromagnetism
  - Number theory
  - Statistics
  - Non-Euclidean geometry
  - Cometary orbital mechanics
  - A founder of the German Magnetic Union
    - Studies the Earth's magnetic field

24.2 Gauss's Law

- Gauss's law is an expression of the general relationship between the net electric flux through a closed surface and the charge enclosed by the surface
  - The closed surface is often called a *gaussian surface*
- Gauss's law is of fundamental importance in the study of electric fields

Gauss's Law – General

- A positive point charge, *q*, is located at the center of a sphere of radius *r*
- The magnitude of the electric field everywhere on the surface of the sphere is
  $E = k_e q / r^2$



Gauss's Law – General, cont.

- The field lines are directed radially outward and are perpendicular to the surface at every point

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E_n dA$$

- This will be the net flux through the gaussian surface, the sphere of radius *r*
- We know $E = k_e q/r^2$ and $A_{sphere} = 4\pi r^2$,

$$\Phi_E = 4\pi k_e q = \frac{q}{\varepsilon_o}$$

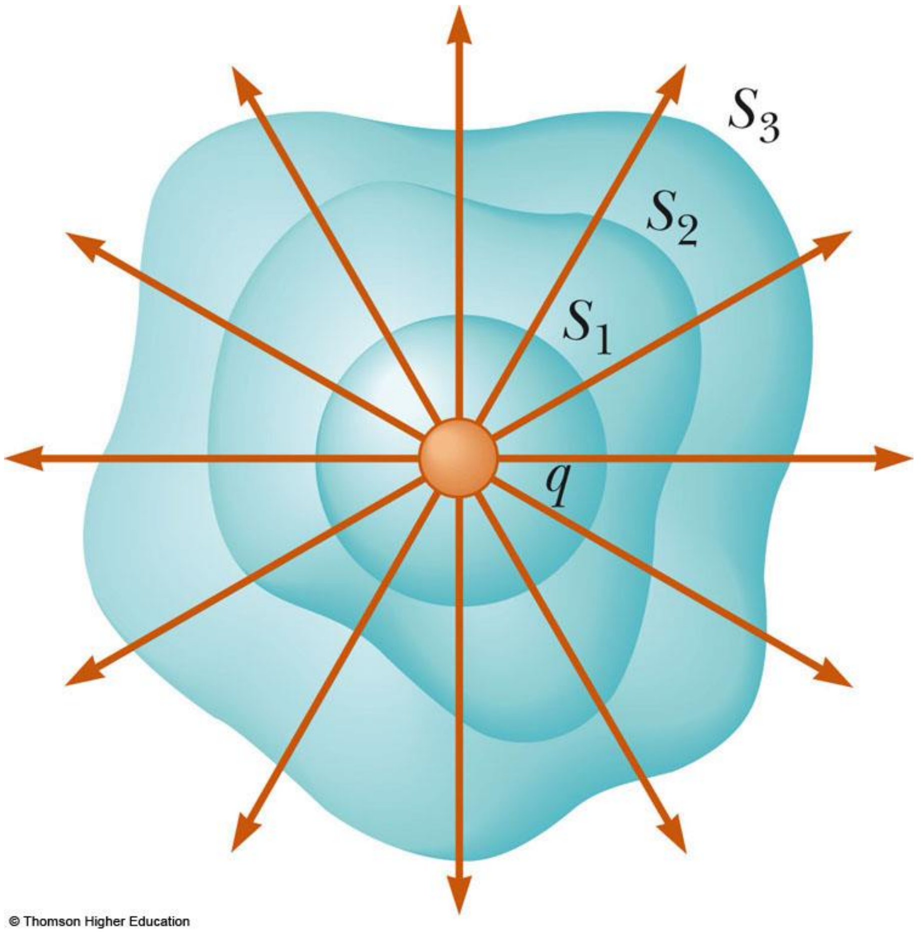
Gauss's Law – General, notes

- The net flux through any closed surface surrounding a point charge, *q*, is given by q/ε_o and is independent of the shape of that surface
- The net electric flux through a closed surface that surrounds no charge is zero
- Since the electric field due to many charges is the vector sum of the electric fields produced by the individual charges, the flux through any closed surface can be expressed as

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint \left(\vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \ldots \right) \cdot d\vec{\mathbf{A}}$$

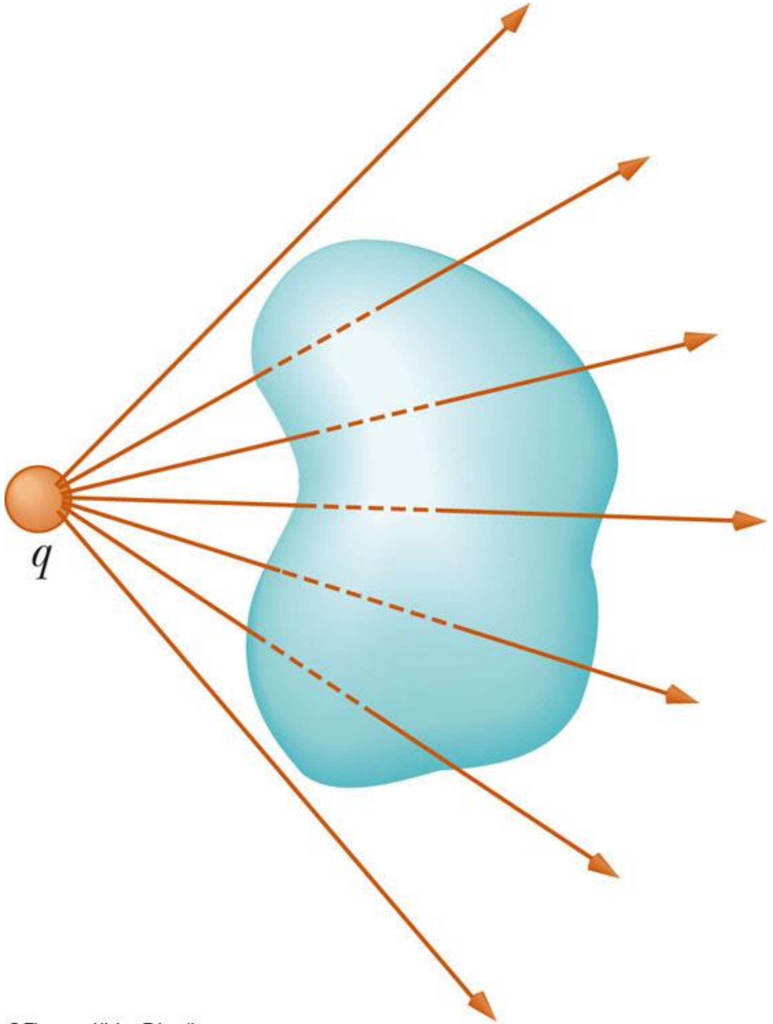
Gaussian Surface, Example

- Closed surfaces of various shapes can surround the charge
  - Only S_1 is spherical
- Verifies the net flux through any closed surface surrounding a point charge q is given by q/ε_o and is independent of the shape of the surface



Gaussian Surface, Example 2

- The charge is outside the closed surface with an arbitrary shape
- Any field line entering the surface leaves at another point
- Verifies the electric flux through a closed surface that surrounds no charge is zero



Gauss's Law – Final

- Gauss's law states $$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\varepsilon_o}$$
  - q_{in} is the net charge inside the surface
- $\vec{\mathbf{E}}$ represents the electric field at any point on the surface
  - $\vec{\mathbf{E}}$ is the *total electric field* and may have contributions from charges both inside and outside of the surface
- Although Gauss's law can, in theory, be solved to find $\vec{\mathbf{E}}$ for any charge configuration, in practice it is limited to symmetric situations

24.3 Application of Gauss's Law

- To use Gauss's law, you want to choose a gaussian surface over which the surface integral can be simplified and the electric field determined
- Take advantage of symmetry
- Remember, the gaussian surface is a surface you choose, it does not have to coincide with a real surface

Conditions for a Gaussian Surface

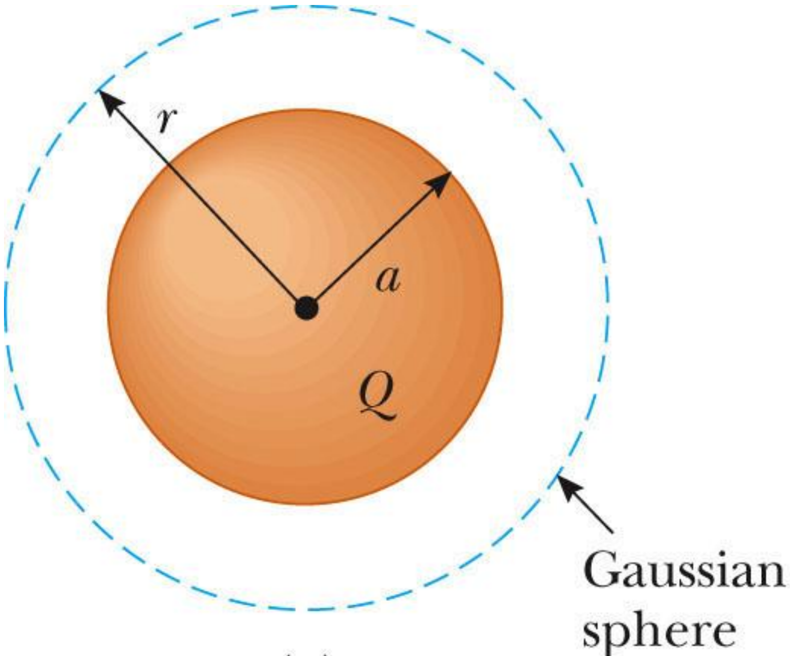
- Try to choose a surface that satisfies one or more of these conditions:
  - The value of the electric field can be argued from symmetry to be constant over the surface
  - The dot product of $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ can be expressed as a simple algebraic product *EdA* because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are parallel
  - The dot product is 0 because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are perpendicular
  - The field is zero over the portion of the surface

Field Due to a Spherically Symmetric Charge Distribution

- Select a sphere as the gaussian surface
- For *r* >*a*

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E_n dA = \frac{q_{in}}{\varepsilon_o}$$

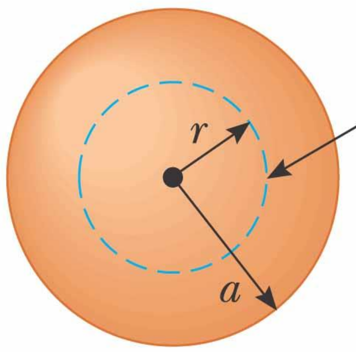
$$E(4\pi r^2) = \frac{Q}{\varepsilon_o} \Rightarrow E = \frac{Q}{4\pi\varepsilon_o r^2} = k_e \frac{Q}{r^2}$$



(a)

Spherically Symmetric, cont.

- Select a sphere as the gaussian surface, $r < a$
- $q_{in} < Q$
- $q_{in} = \rho(4/3\pi r^3)$
- $\rho = Q/\ (4/3\pi a^3)$



(b)

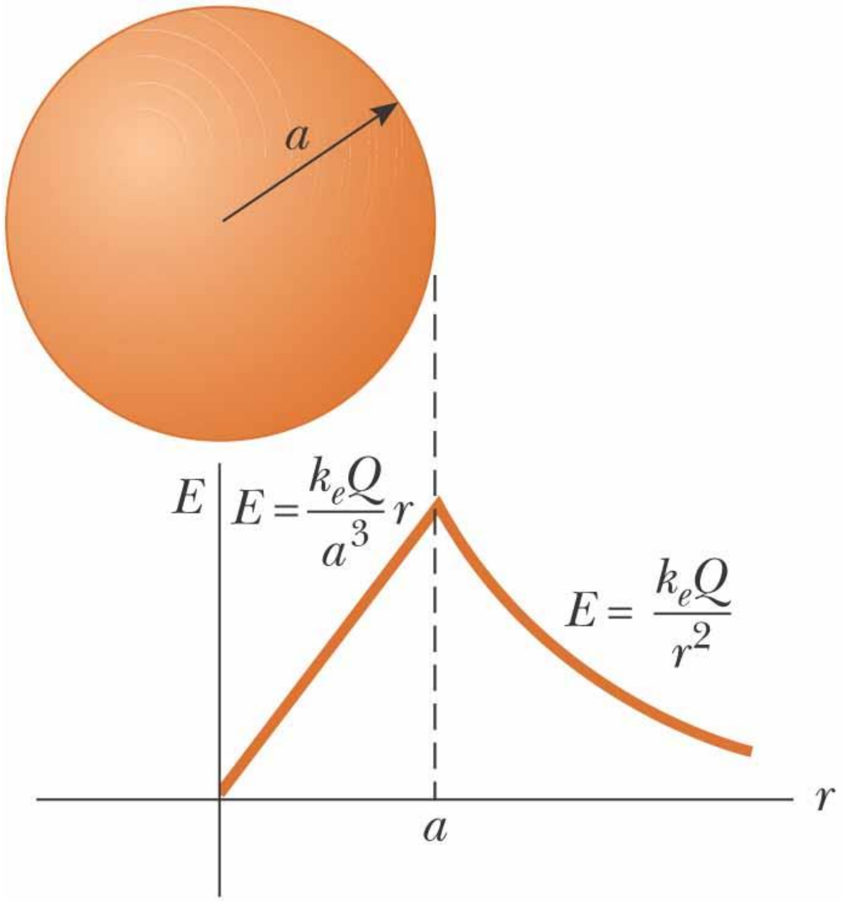
$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E_n dA = \frac{q_{in}}{\varepsilon_o}$$

$$E(4\pi r^2) = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{\varepsilon_o} \Rightarrow$$

$$E = \frac{\rho}{3\varepsilon_o} r = \frac{Q/\left(\frac{4}{3}\pi a^3\right)}{3\varepsilon_o} r = k_e \frac{Q}{a^3} r$$

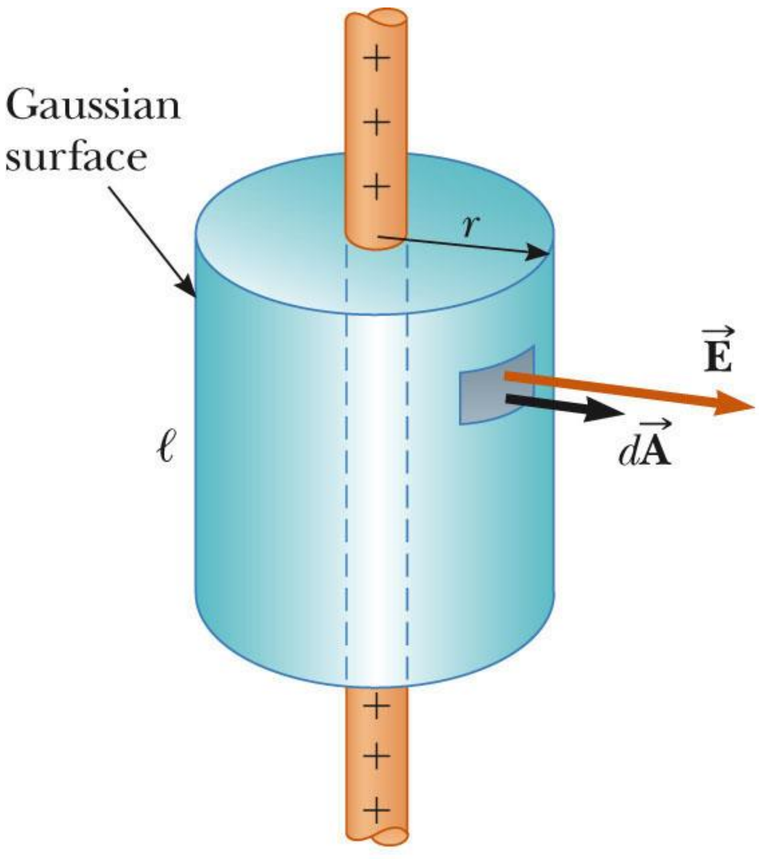
Spherically Symmetric Distribution, final

- Inside the sphere, E varies linearly with r
  - $E \to 0$ as $r \to 0$
- The field outside the sphere is equivalent to that of a point charge located at the center of the sphere



Field at a Distance from a Line of Charge

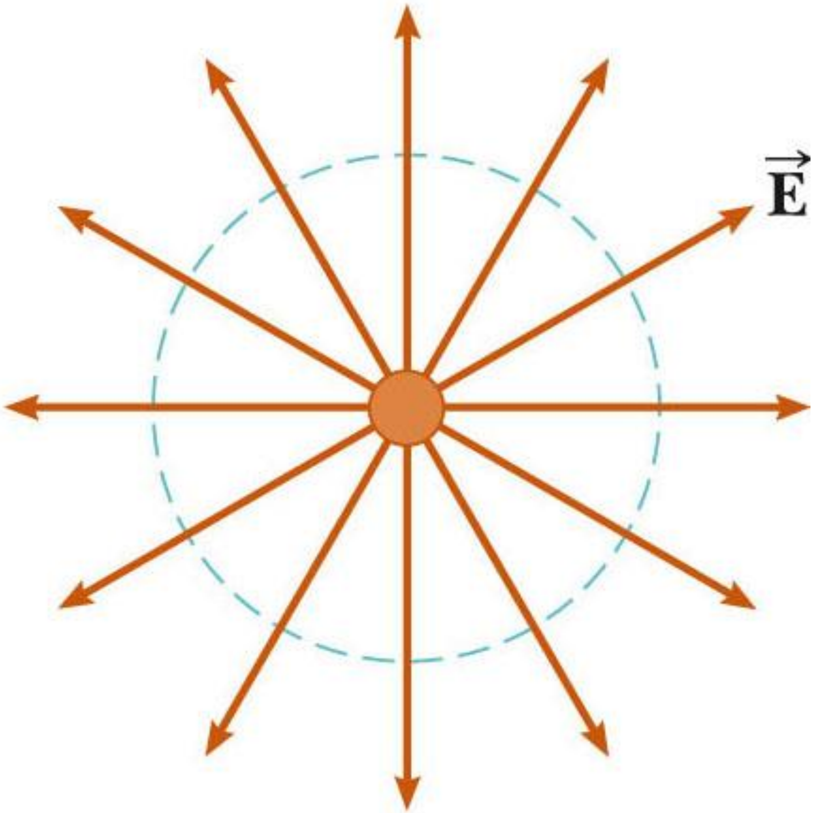
- Select a cylindrical charge distribution
  - The cylinder has a radius of *r* and a length of *ℓ*
- $\vec{\mathbf{E}}$ is constant in magnitude and perpendicular to the surface at every point on the curved part of the surface



(a)

Field Due to a Line of Charge, cont.

- The end view confirms the field is perpendicular to the curved surface
- The field through the ends of the cylinder is 0 since the field is parallel to these surfaces



(b)

Field Due to a Line of Charge, final

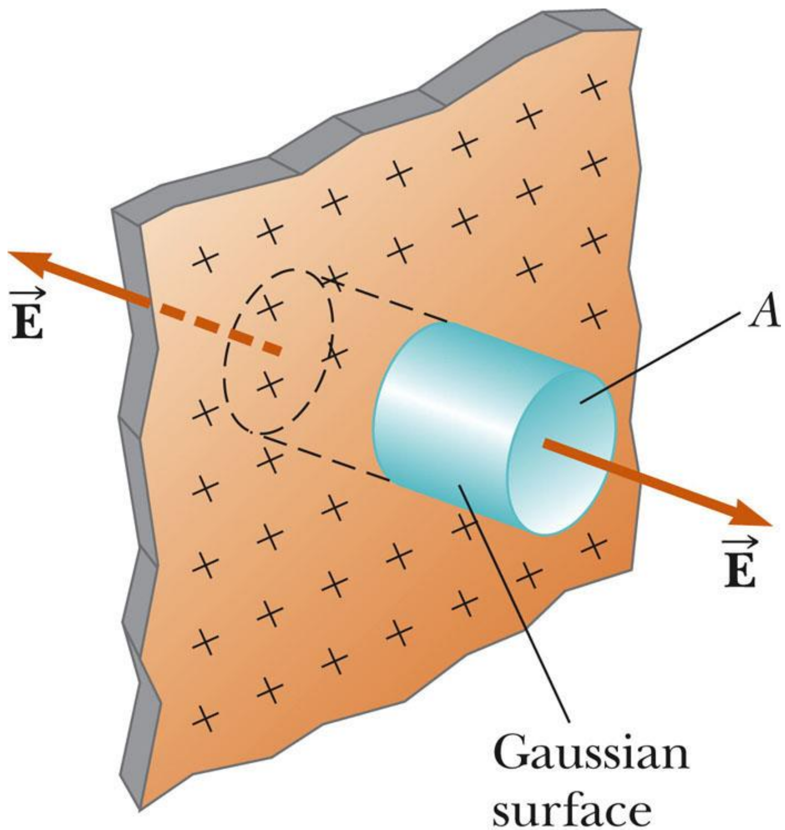
- Use Gauss's law to find the field

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = \oint E_n dA = \frac{q_{in}}{\varepsilon_o}$$

$$E(2\pi rl) = \frac{\lambda l}{\varepsilon_o} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_o r} = 2k_e \frac{\lambda}{r}$$

Field Due to a Plane of Charge

- $\vec{\mathbf{E}}$ must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane
- Choose a small cylinder whose axis is perpendicular to the plane for the gaussian surface



Field Due to a Plane of Charge, cont

- $\vec{\mathbf{E}}$ is parallel to the curved surface and there is no contribution to the surface area from this curved part of the cylinder
- The flux through each end of the cylinder is *EA* and so the total flux is 2*EA*

Field Due to a Plane of Charge, final

- The total charge in the surface is σA
- Applying Gauss's law

$$\Phi_E = 2EA = \frac{\sigma A}{\varepsilon_o} \text{ and } E = \frac{\sigma}{2\varepsilon_o}$$

- Note, this does not depend on r
- Therefore, the field is uniform everywhere

Electrostatic Equilibrium

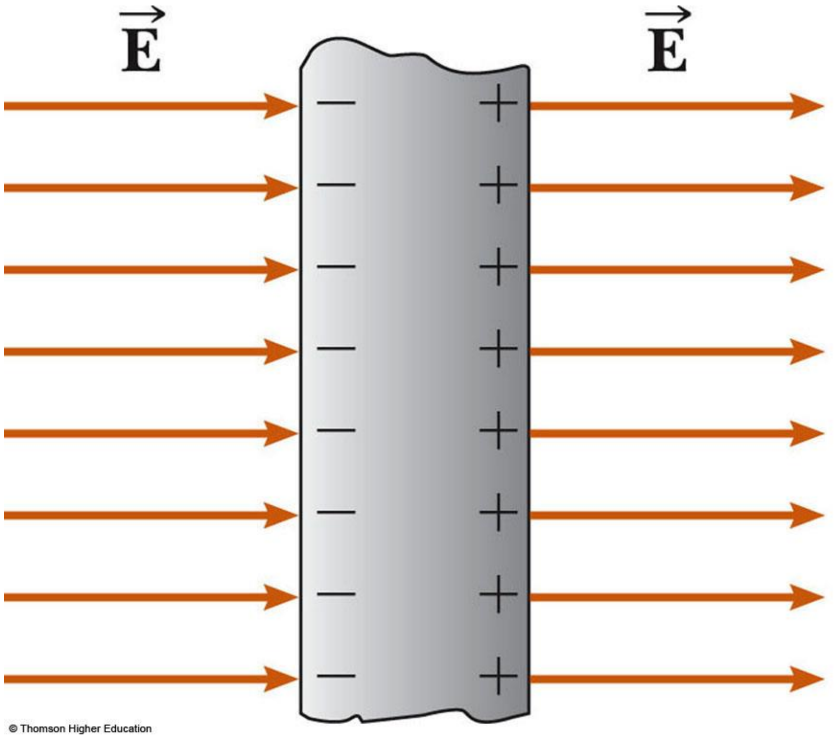
- When there is no net motion of charge within a conductor, the conductor is said to be in **electrostatic equilibrium**

24.4 Conductors in Electrostatic Equilibrium

- The electric field is zero everywhere inside the conductor
  - Whether the conductor is solid or hollow
- If an isolated conductor carries a charge, the charge resides on its surface
- The electric field just outside a charged conductor is perpendicular to the surface and has a magnitude of σ/ε_o
  - σ is the surface charge density at that point
- On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature is the smallest

Property 1: $\text{Field}_{\text{inside}} = 0$

- Consider a conducting slab in an external field $\vec{\mathbf{E}}$
- If the field inside the conductor were not zero, free electrons in the conductor would experience an electrical force
- These electrons would accelerate
- These electrons would not be in equilibrium
- Therefore, there cannot be a field inside the conductor

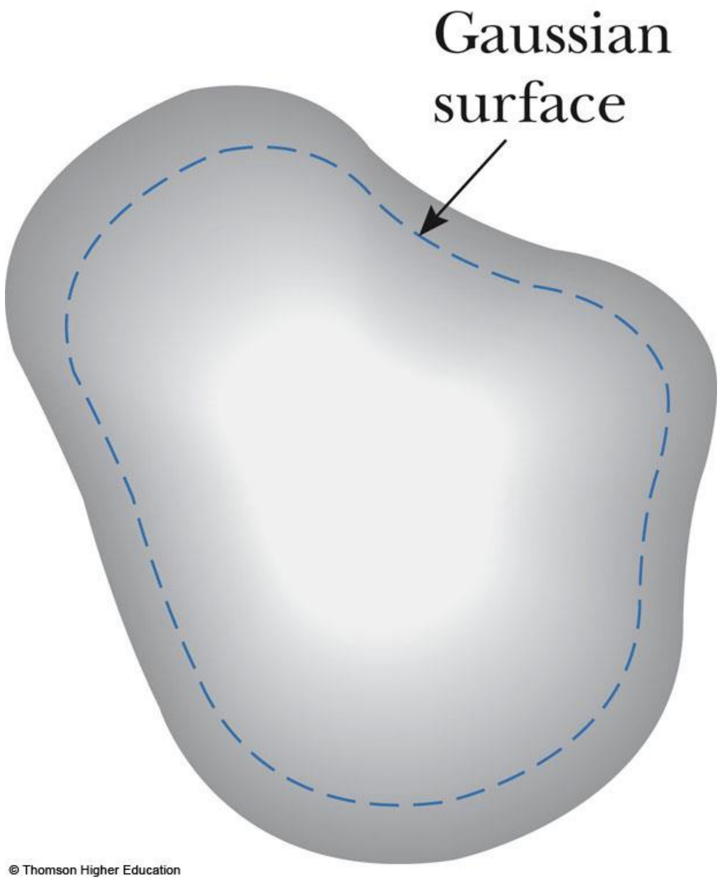


Property 1: $\text{Field}_{\text{inside}} = 0$, cont.

- Before the external field is applied, free electrons are distributed throughout the conductor
- When the external field is applied, the electrons redistribute until the magnitude of the internal field equals the magnitude of the external field
- There is a net field of zero inside the conductor
- This redistribution takes about 10^{-16}s and can be considered instantaneous

Property 2: Charge Resides on the Surface

- Choose a gaussian surface inside but close to the actual surface
- The electric field inside is zero (prop. 1)
- There is no net flux through the gaussian surface
- Because the gaussian surface can be as close to the actual surface as desired, there can be no charge inside the surface

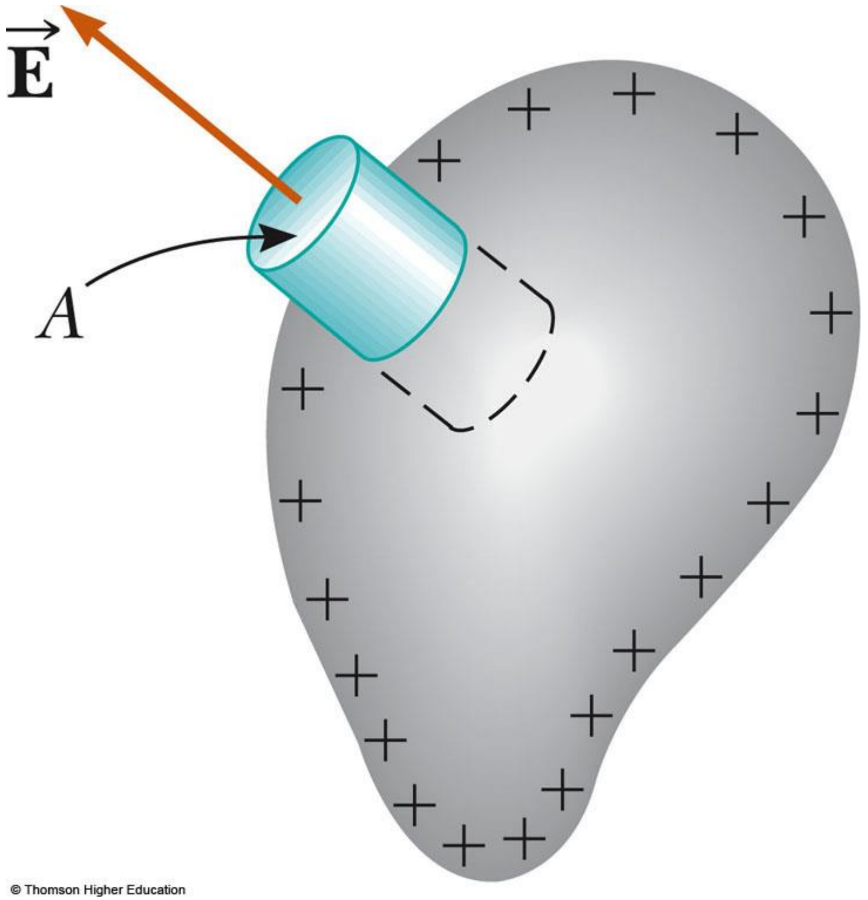


Property 2: Charge Resides on the Surface, cont

- Since no net charge can be inside the surface, any net charge must reside ***on*** the surface
- Gauss's law does not indicate the distribution of these charges, only that it must be on the surface of the conductor

Property 3: Field's Magnitude and Direction

- Choose a cylinder as the gaussian surface
- The field must be perpendicular to the surface
  - If there were a parallel component to $\vec{\mathbf{E}}$, charges would experience a force and accelerate along the surface and it would not be in equilibrium



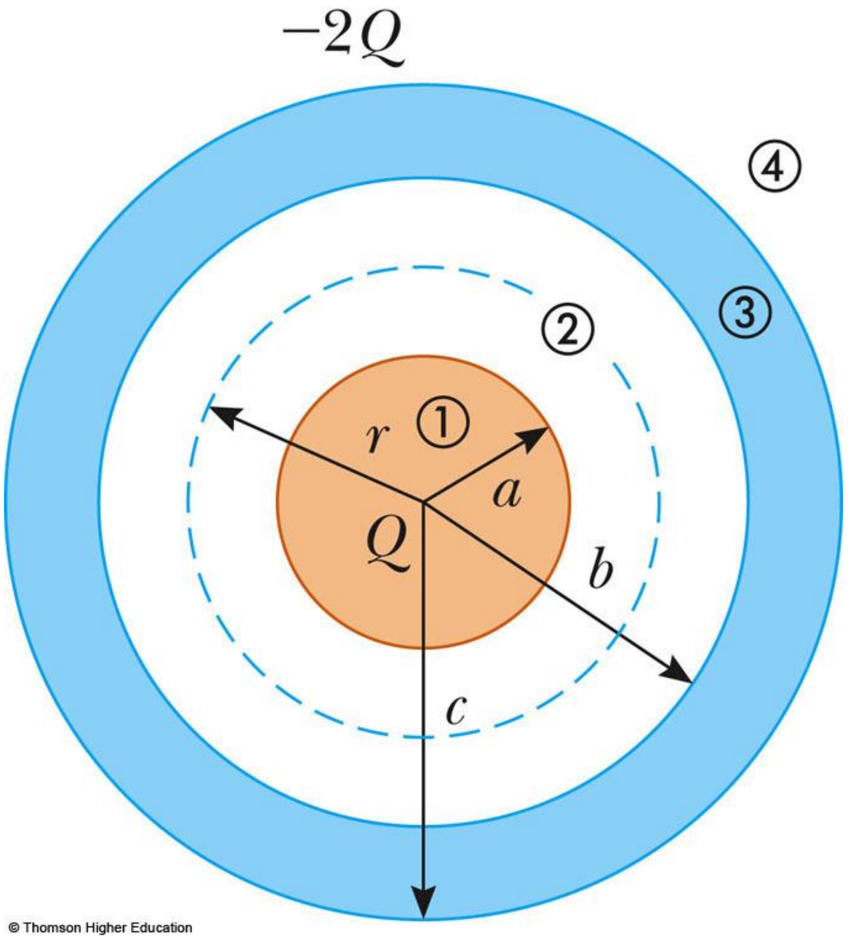
Property 3: Field's Magnitude and Direction, cont.

- The net flux through the gaussian surface is through only the flat face outside the conductor
  - The field here is perpendicular to the surface
- Applying Gauss's law

$$\Phi_E = EA = \frac{\sigma A}{\varepsilon_o} \text{ and } E = \frac{\sigma}{\varepsilon_o}$$

Sphere and Shell Example

- *Conceptualize*
  - Similar to the sphere example
  - Now a charged sphere is surrounded by a shell
  - Note charges
- *Categorize*
  - System has spherical symmetry
  - Gauss' Law can be applied



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Sphere and Shell Example

- *Analyze*
  - Construct a Gaussian sphere between the surface of the solid sphere and the inner surface of the shell
  - The electric field lines must be directed radially outward and be constant in magnitude on the Gaussian surface

Sphere and Shell Example, 3

- *Analyze, cont*
  - The electric field for each area can be calculated

$$E_1 = k_e \frac{Q}{a^3} r \quad (for\ r < a)$$

$$E_2 = k_e \frac{Q}{r^2} \quad (for\ a < r < b)$$

$$E_3 = 0 \quad (for\ b < r < c)$$

$$E_4 = -k_e \frac{Q}{r^2} \quad (for\ r > c)$$

Sphere and Shell Example

- *Finalize*
  - Check the net charge
  - Think about other possible combinations
    - What if the sphere were conducting instead of insulating?

Engineering Electromagnetics

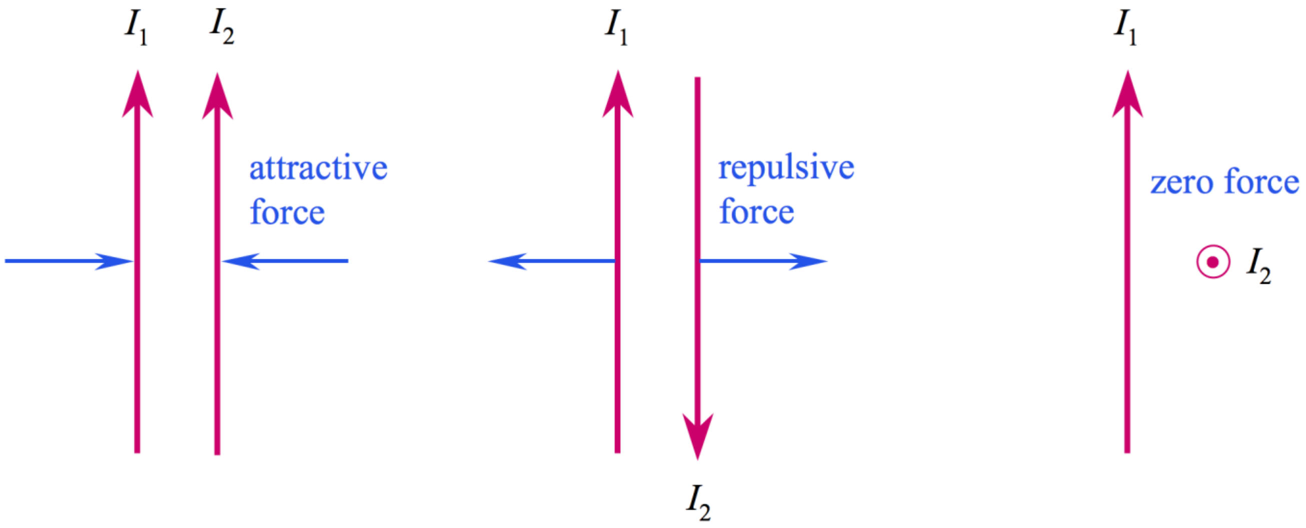
Chapter 7:

The Steady Magnetic Field

Motivating the Magnetic Field Concept: Forces Between Currents

Magnetic forces arise whenever we have charges in motion. Forces between current-carrying wires present familiar examples that we can use to determine what a magnetic force field should look like:

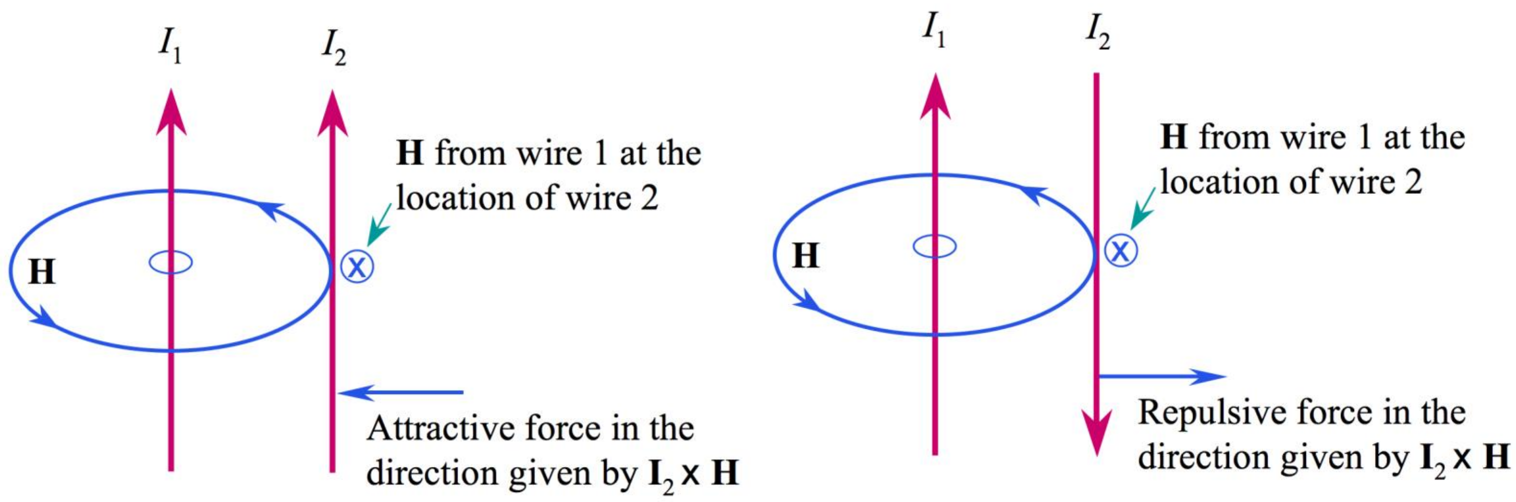
Here are the easily-observed facts:



How can we describe a force field around wire 1 that can be used to determine the force on wire 2?

Magnetic Field

The geometry of the magnetic field is set up to correctly model forces between currents that allow for any relative orientation. The magnetic field intensity, **H**, circulates *around* its source, I_1, in a direction most easily determined by the *right-hand rule*: Right thumb in the direction of the current, fingers curl in the direction of **H**



Note that in the third case (perpendicular currents), I_2 is in the same direction as **H**, so that their cross product (and the resulting force) is zero. The actual force computation involves a different field quantity, **B**, which is related to **H** through $\mathbf{B} = \mu_0\mathbf{H}$ in free space. This will be taken up in a later lecture. Our immediate concern is how to find **H** from any given current distribution.

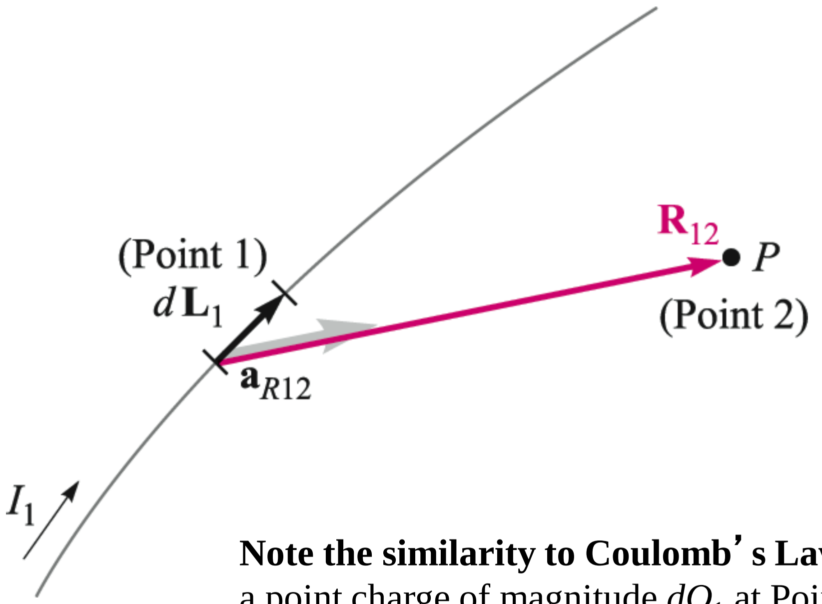
Biot-Savart Law

The Biot-Savart Law specifies the magnetic field intensity, **H**, arising from a "point source" current element of differential length d**L.**

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

The units of **H** are [A/m]

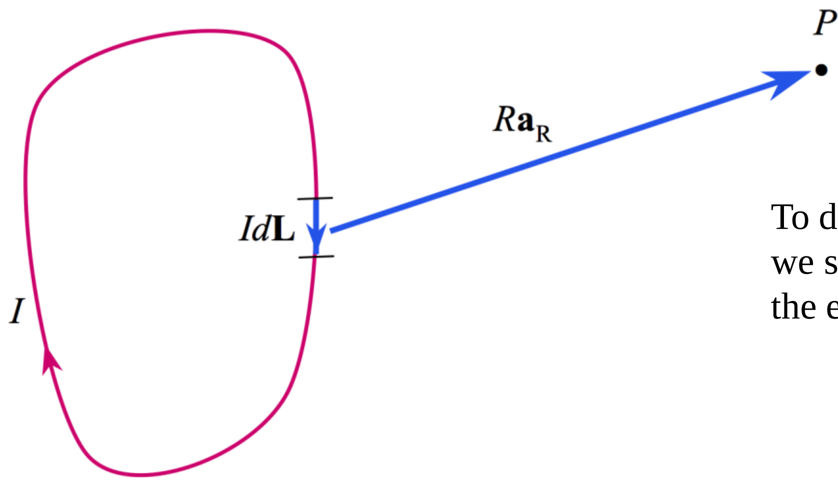
Note in particular the inverse-square distance dependence, and the fact that the cross product will yield a field vector that points into the page. This is a formal statement of the right-hand rule



Note the similarity to Coulomb's Law, in which a point charge of magnitude dQ_1 at Point 1 would generate electric field at Point 2 given by:

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi \epsilon_0 R_{12}^2}$$

Magnetic Field Arising From a Circulating Current



At point *P*, the magnetic field associated with the differential current element $Id\mathbf{L}$ is

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

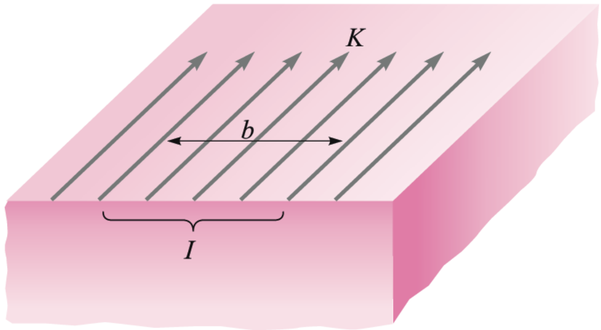
To determine the total field arising from the closed circuit path, we sum the contributions from the current elements that make up the entire loop, or

$$\boxed{\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

The contribution to the field at *P* from any portion of the current will be just the above integral evalated over just that portion.

Two- and Three-Dimensional Currents

On a surface that carries uniform surface current density **K** [A/m], the current within width *b* is



$$I = Kb$$

..and so the differential current quantity that appears in the Biot-Savart law becomes:

$$I\,d\mathbf{L} = \mathbf{K}\,dS$$

The magnetic field arising from a current sheet is thus found from the two-dimensional form of the Biot-Savart law:

$$\boxed{\mathbf{H} = \int_S \frac{\mathbf{K} \times \mathbf{a}_R\,dS}{4\pi R^2}}$$

In a similar way, a **volume current** will be made up of three-dimensional current elements, and so the Biot-Savart law for this case becomes:

$$\boxed{\mathbf{H} = \int_{\mathrm{vol}} \frac{\mathbf{J} \times \mathbf{a}_R\,dv}{4\pi R^2}}$$

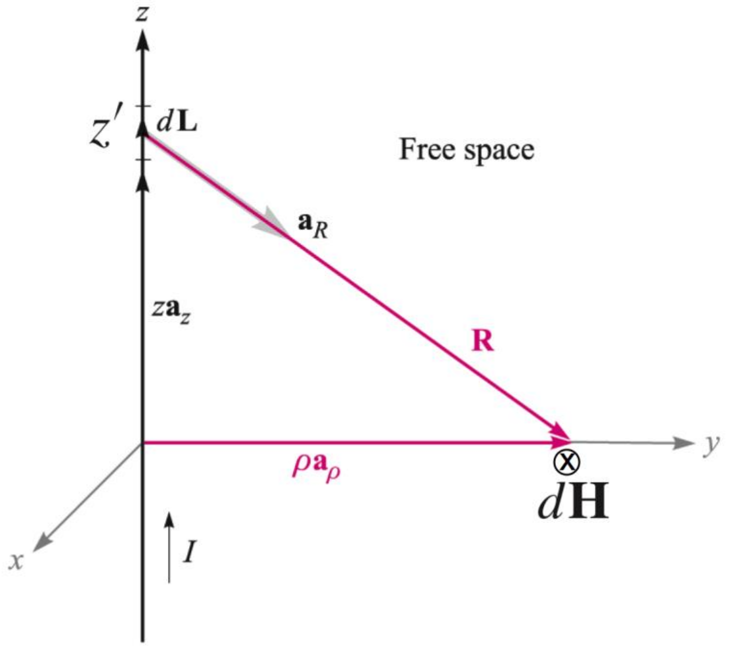
Example of the Biot-Savart Law

In this example, we evaluate the magnetic field intensity on the *y* axis (equivalently in the *xy* plane) arising from a filament current of infinite length in on the *z* axis.

Using the drawing, we identify:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

and so.. $$\mathbf{a}_R = \frac{\rho\mathbf{a}_\rho - z'\mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$



so that:

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

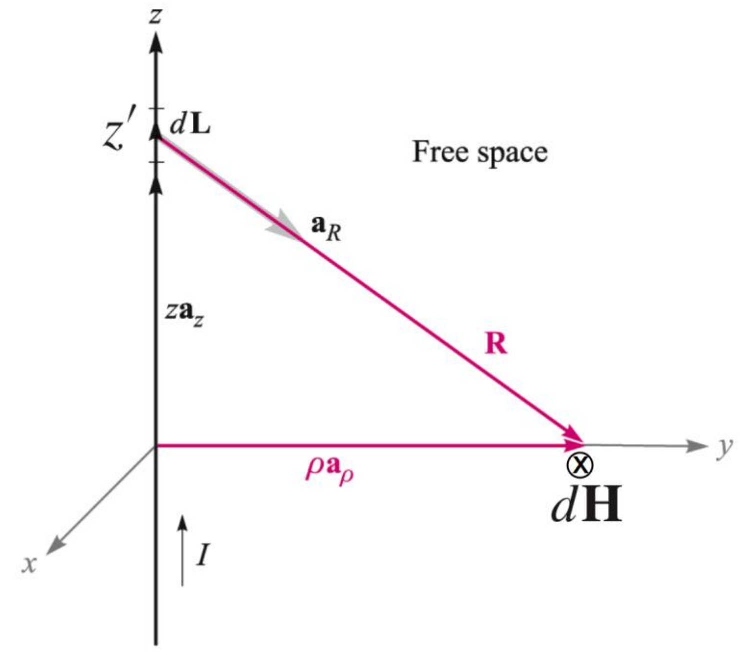
Example: continued

We now have: $$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Integrate this over the entire wire:

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\,dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$



..after carrying out the cross product

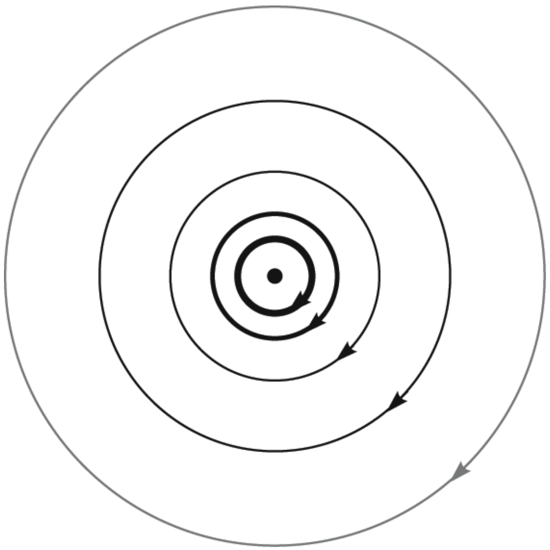
Example: concluded

Evaluating the integral:

we have: $$\mathbf{H} = \frac{I}{4\pi}\int_{-\infty}^{\infty}\frac{\rho\, dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I\rho\mathbf{a}_\phi}{4\pi}\left.\frac{z'}{\rho^2\sqrt{\rho^2 + z'^{\,2}}}\right|_{-\infty}^{\infty}$$

finally: $$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi}$$



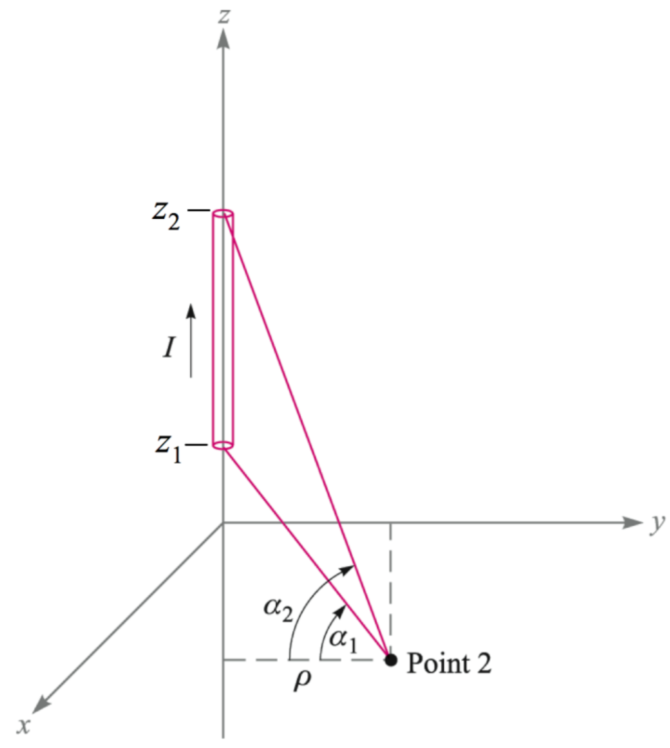
Current is into the page.
Magnetic field streamlines
are concentric circles, whose magnitudes
decrease as the inverse distance from the z axis

Field Arising from a Finite Current Segment

In this case, the field is to be found in the *xy* plane at Point 2.
The Biot-Savart integral is taken over the wire length:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$$= \int_{\rho \tan \alpha_1}^{\rho \tan \alpha_2} \frac{I dz \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z \mathbf{a}_z)}{4\pi(\rho^2 + z^2)^{3/2}}$$



..after a few additional steps (see Problem 7.8), we find:

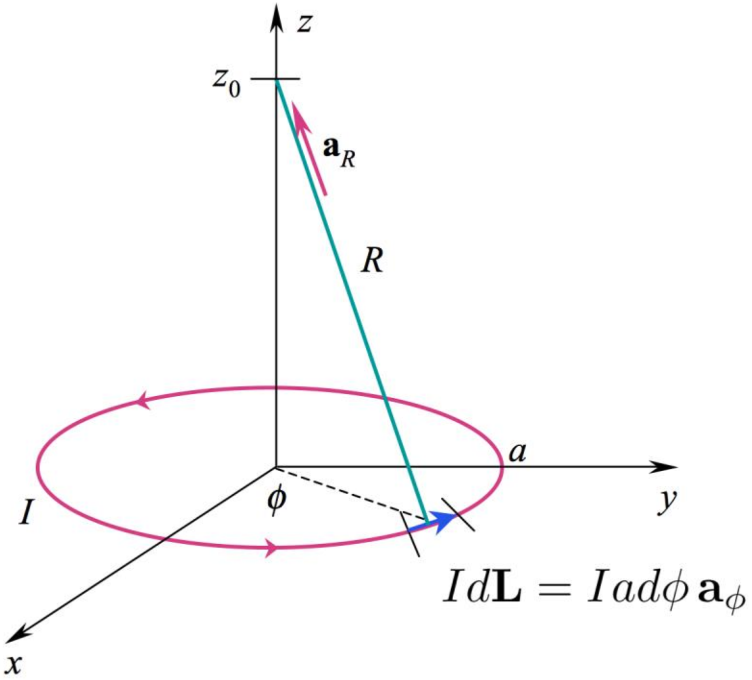
$$\boxed{\mathbf{H} = \frac{I}{4\pi\rho}(\sin \alpha_2 - \sin \alpha_1)\mathbf{a}_\phi}$$

Another Example: Magnetic Field from a Current Loop

Consider a circular current loop of radius *a* in the *x*-*y* plane, which carries steady current *I*. We wish to find the magnetic field strength anywhere on the *z* axis.

We will use the Biot-Savart Law:

$$\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$



where: $$Id\mathbf{L} = Iad\phi\,\mathbf{a}_\phi$$

$$R = \sqrt{a^2 + z_0^2}$$

$$\mathbf{a}_R = \frac{z_0\,\mathbf{a}_z - a\,\mathbf{a}_\rho}{\sqrt{a^2 + z_0^2}}$$

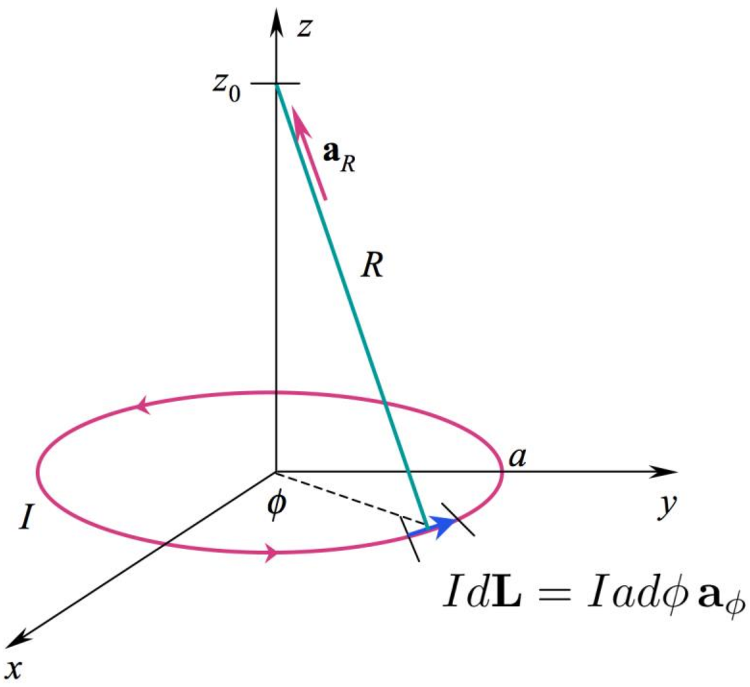
Example: Continued

Substituting the previous expressions, the Biot-Savart Law becomes:

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times (z_0\,\mathbf{a}_z - a\,\mathbf{a}_\rho)}{4\pi(a^2 + z_0^2)^{3/2}}$$

carry out the cross products to find:

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,(z_0\,\mathbf{a}_\rho + a\,\mathbf{a}_z)}{4\pi(a^2 + z_0^2)^{3/2}}$$



but we must include the angle dependence in the radial unit vector:

$$\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$$

with this substitution, the radial component will integrate to zero, meaning that all radial components will cancel on the *z* axis.

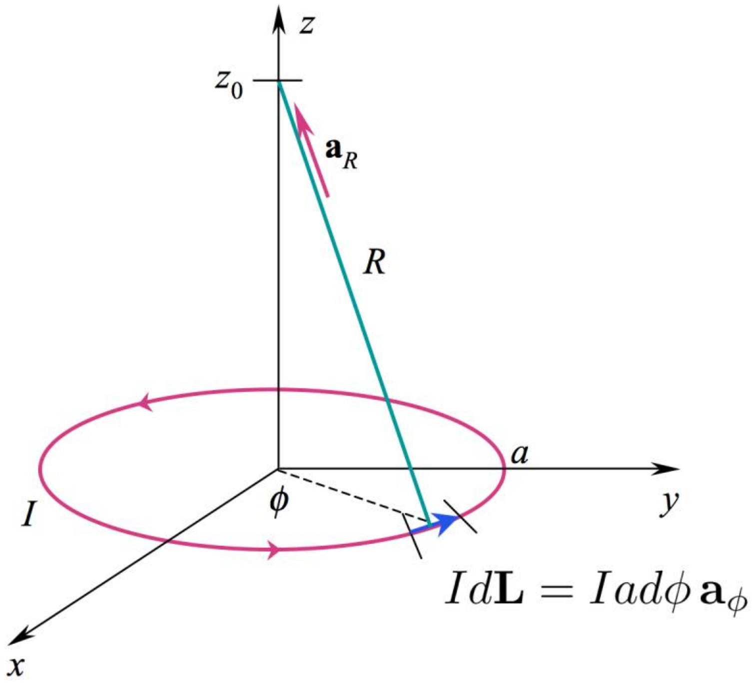
Example: Continued

Now, only the z component remains, and the integral evaluates easily:

$$\boxed{\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}}$$

Note the form of the numerator: the product of the current and the loop area. We define this as the *magnetic moment*:

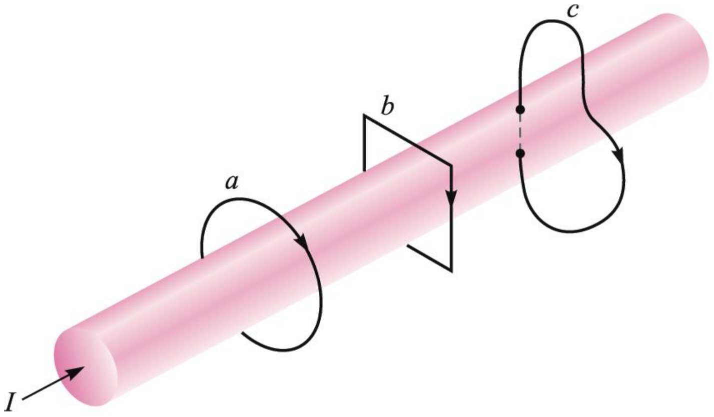
$$\mathbf{m} = I(\pi a^2)\,\mathbf{a}_z$$



Ampere's Circuital Law

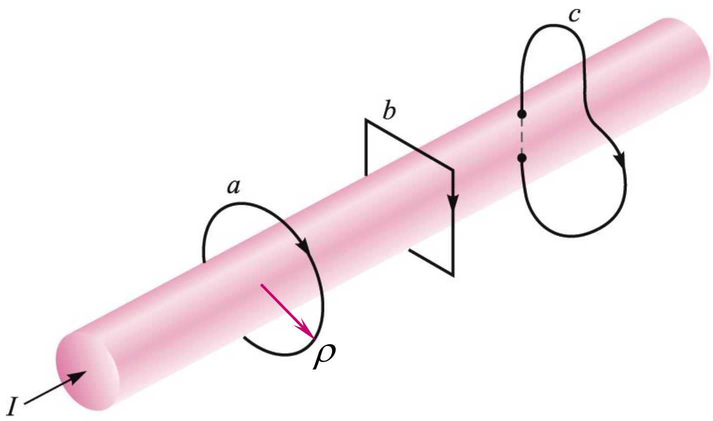
Ampere's Circuital Law states that the line integral of **H** about *any closed path* is exactly equal to the direct current enclosed by that path.

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$



In the figure at right, the integral of **H** about closed paths *a* and *b* gives the total current *I*, while the integral over path *c* gives only that portion of the current that lies within *c*

Ampere's Law Applied to a Long Wire



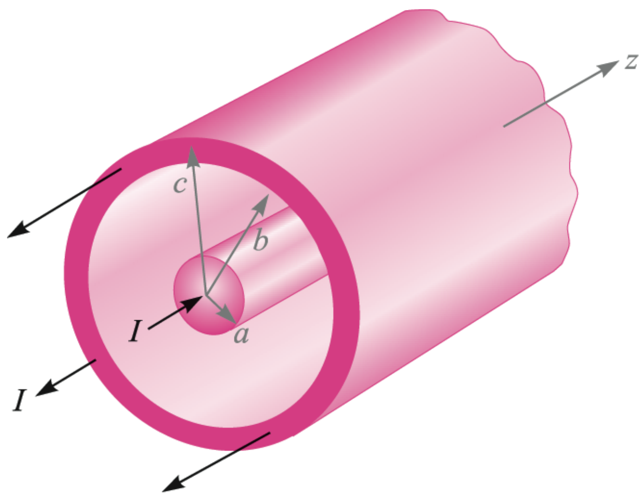
Symmetry suggests that **H** will be circular, constant-valued at constant radius, and centered on the current (*z*) axis.

Choosing path *a*, and integrating **H** around the circle of radius ρ gives the enclosed current, *I*:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho \, d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I$$

so that: $\boxed{H_\phi = \dfrac{I}{2\pi\rho}}$ as before.

Coaxial Transmission Line



In the coax line, we have two concentric *solid* conductors that carry equal and opposite currents, I.

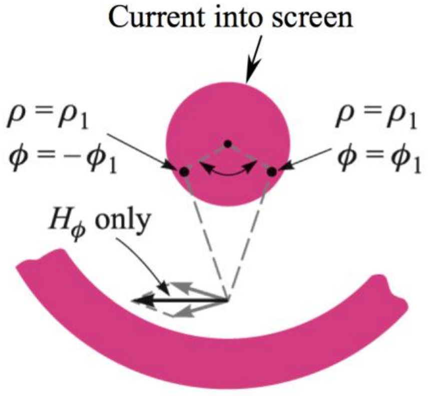
The line is assumed to be infinitely long, and the circular symmetry suggests that **H** will be entirely ϕ - directed, and will vary only with radius ρ.

Our objective is to find the magnetic field for all values of ρ

Field Between Conductors

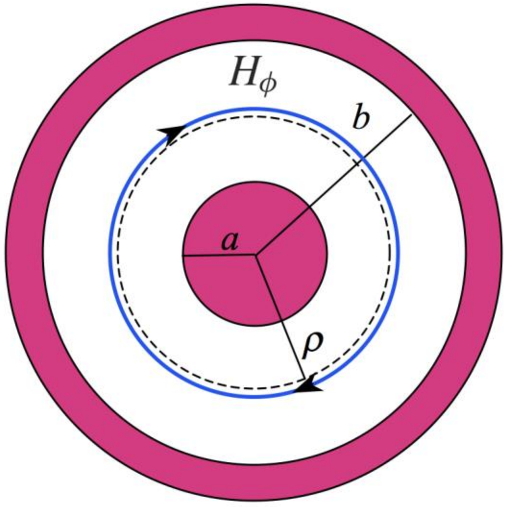
The inner conductor can be thought of as made up of a bundle of filament currents, each of which produces the field of a long wire.

Consider two such filaments, located at the same radius from the *z* axis, ρ_1, but which lie at symmetric ϕ coordinates, ϕ_1 and -ϕ_1.Their field contributions superpose to give a net H_ϕ component as shown. The same happens for every pair of symmetrically-located filaments, which taken as a whole, make up the entire center conductor.



The field between conductors is thus found to be the same as that of filament conductor on the *z* axis that carries current, *I*. Specifically:

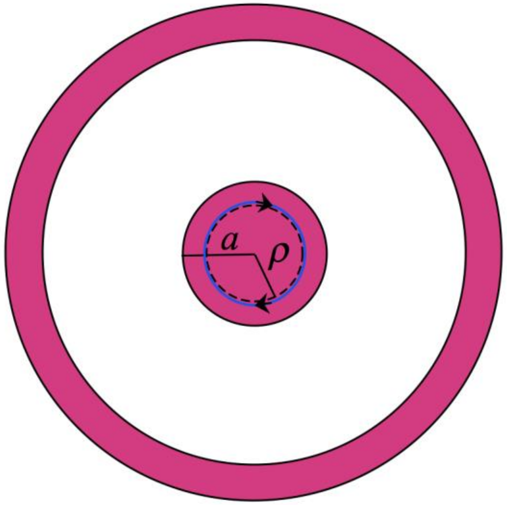
$$H_\phi = \frac{I}{2\pi\rho} \qquad a < \rho < b$$



Field Within the Inner Conductor

With current uniformly distributed inside the conductors, the **H** can be assumed circular everywhere.

Inside the inner conductor, and at radius ρ, we again have:



$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi \ = H_\phi 2\pi \rho$$

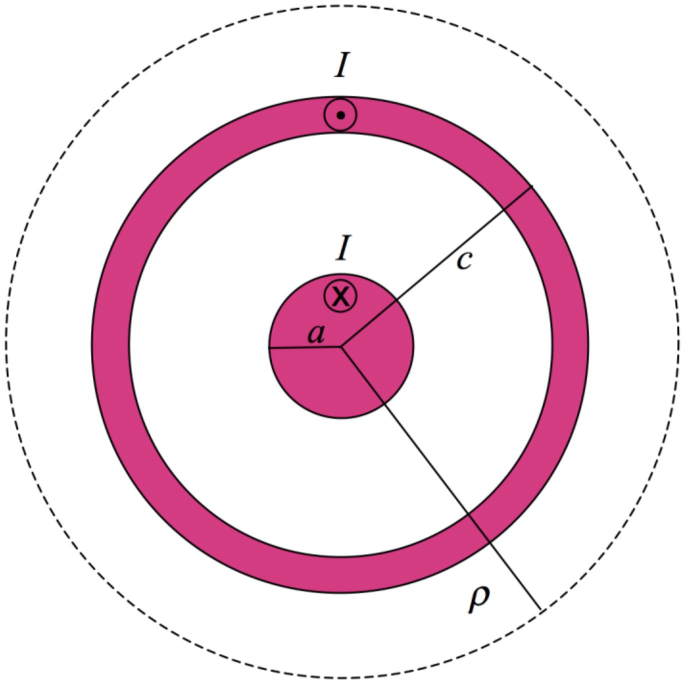
But now, the current enclosed is $I_{\text{encl}} = I \dfrac{\rho^2}{a^2}$

so that $2\pi \rho H_\phi = I \dfrac{\rho^2}{a^2}$ or finally: $\boxed{H_\phi = \dfrac{I\rho}{2\pi a^2} \quad (\rho < a)}$

Field Outside Both Conducors

Outside the transmission line, where $\rho > c$,
no current is enclosed by the integration path,
and so

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0$$

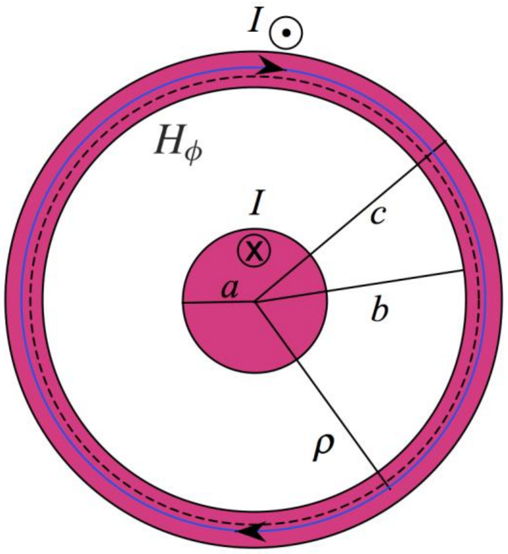


As the current is uniformly distributed, and since we have circular symmetry, the field would have to be constant over the circular integration path, and so it must be true that:

$$\boxed{H_\phi = 0 \quad (\rho > c)}$$

Field Inside the Outer Conductor

Inside the outer conductor, the enclosed current consists of that within the inner conductor plus that portion of the outer conductor current existing at radii less than ρ



Ampere's Circuital Law becomes

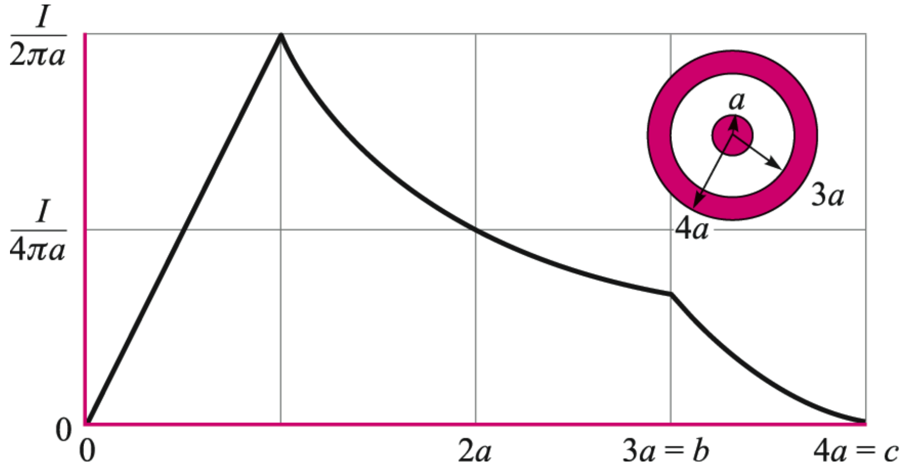
$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

..and so finally:

$$\boxed{H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \quad (b < \rho < c)}$$

Magnetic Field Strength as a Function of Radius in the Coax Line

Combining the previous results, and assigning dimensions as shown in the inset below, we find:



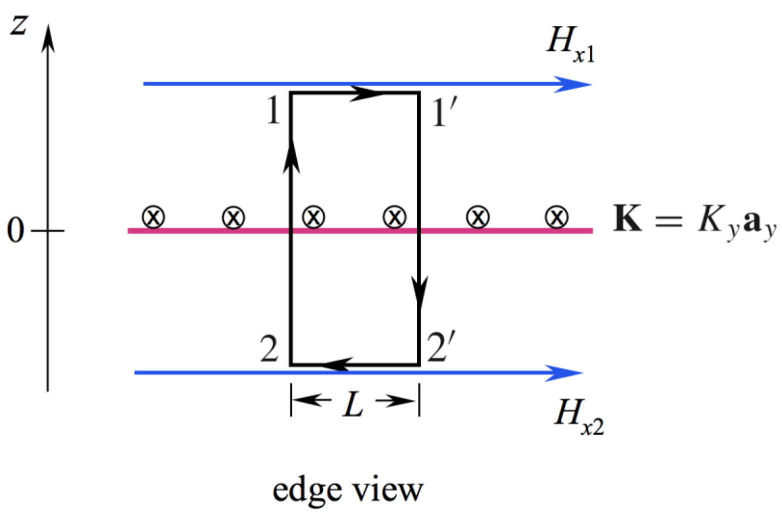
Magnetic Field Arising from a Current Sheet

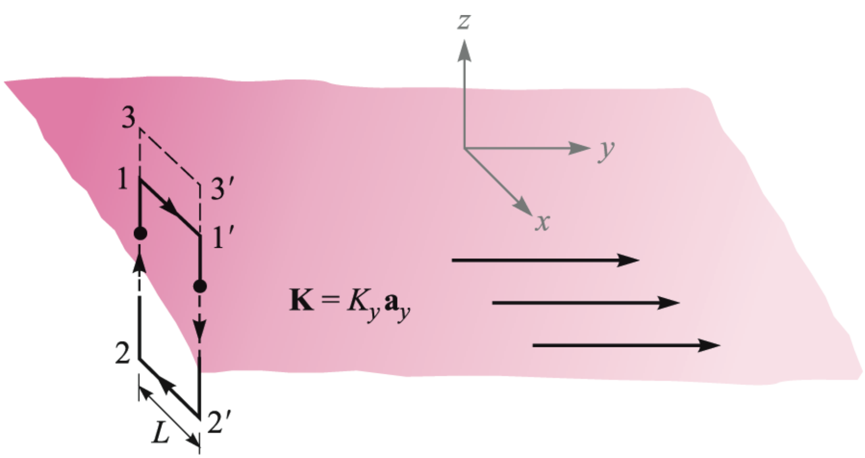
For a uniform plane current in the y direction, we expect an x-directed **H** field from symmetry.

Applying Ampere's circuital law to the path $1\text{-}1'\text{-}2'\text{-}2\text{-}1$ we find:

$$H_{x1}L + H_{x2}(-L) = K_y L \quad \text{or} \quad \underline{H_{x1} - H_{x2} = K_y}$$

In other words, the magnetic field is discontinuous across the current sheet by the magnitude of the surface current density.



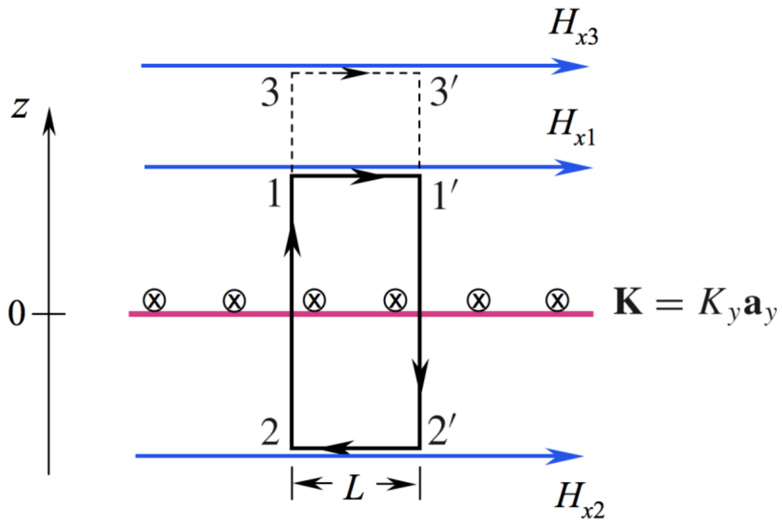


Magnetic Field Arising from a Current Sheet

If instead, the upper path is elevated to the line between 3 and $3'$, the same current is enclosed and we would have

$$H_{x3} - H_{x2} = K_y$$ from which we conclude that $$H_{x3} = H_{x1}$$

so the field is *constant in each region* (above and below the current plane)



edge view

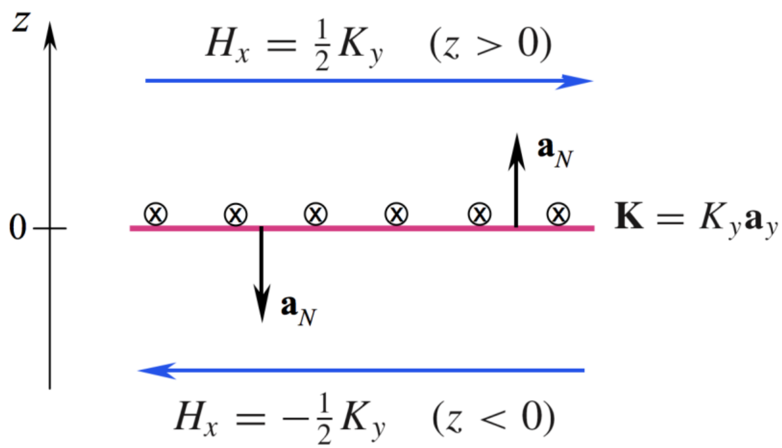
By symmetry, the field above the sheet must be the same in magnitude as the field below the sheet. Therefore, we may state that

$$\boxed{H_x = \tfrac{1}{2}K_y \quad (z > 0)}$$

and $$\boxed{H_x = -\tfrac{1}{2}K_y \quad (z < 0)}$$

Magnetic Field Arising from a Current Sheet

The actual field configuration is shown below, in which magnetic field above the current sheet is equal in magnitude, but in the direction opposite to the field below the sheet.



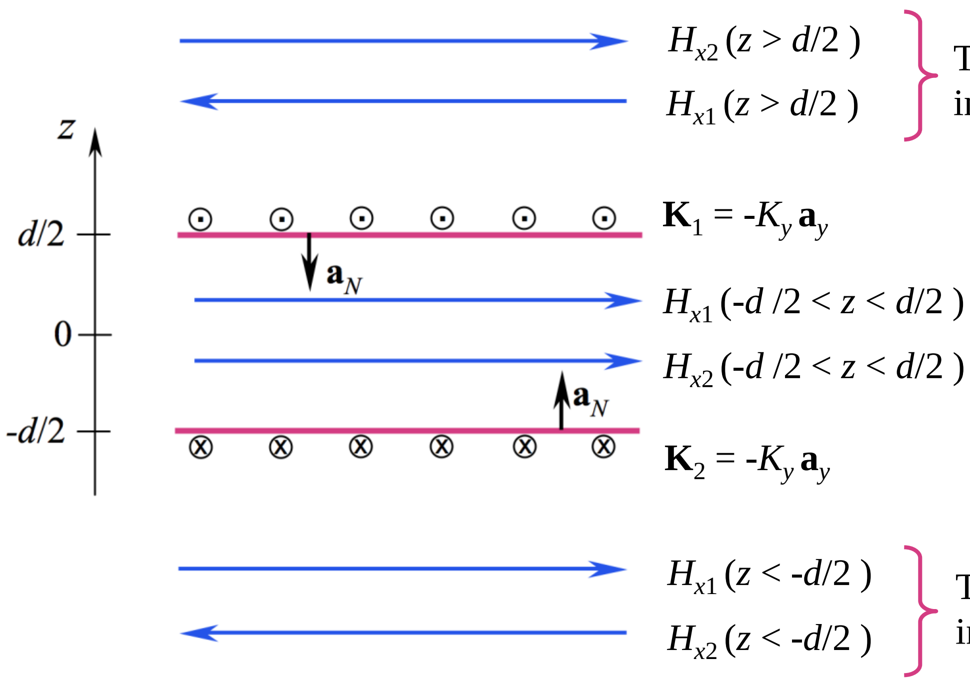
The field in either region is found by the cross product:

$$\boxed{\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N}$$

where $\mathbf{a}_N$ is the unit vector that is normal to the current sheet, and that points into the region in which the magnetic field is to be evaluated.

Magnetic Field Arising from Two Current Sheets

Here are two parallel currents, equal and opposite, as you would find in a parallel-plate transmission line. If the sheets are much wider than their spacing, then the magnetic field will be contained in the region between plates, and will be nearly zero outside.



These fields cancel for current sheets of infinite width.

These fields are equal and add to give

$$\boxed{\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (-d/2 < z < d/2\,)}$$

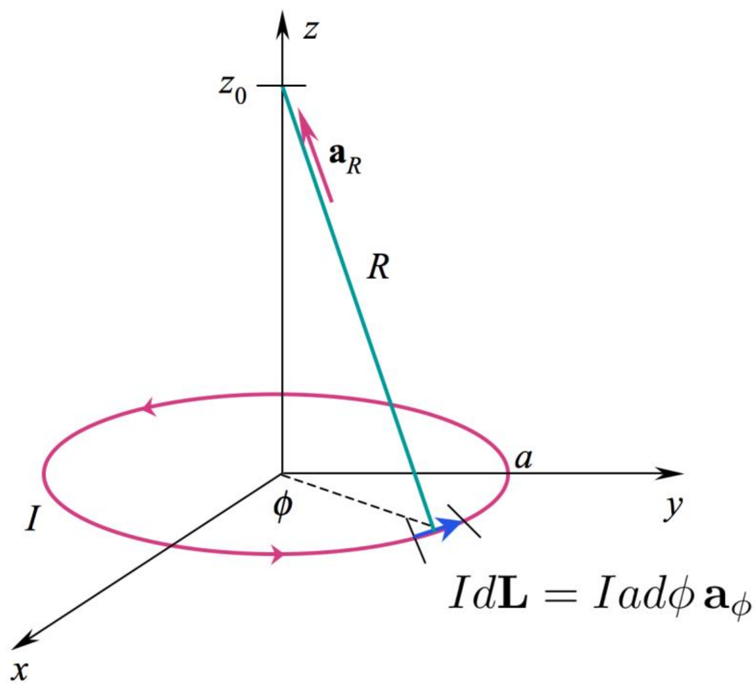
where $\mathbf{K}$ is either $\mathbf{K}_1$ or $\mathbf{K}_2$

These fields cancel for current sheets of infinite width.

Current Loop Field

Using the Biot-Savart Law, we previously found the magnetic field on the *z* axis from a circular current loop:

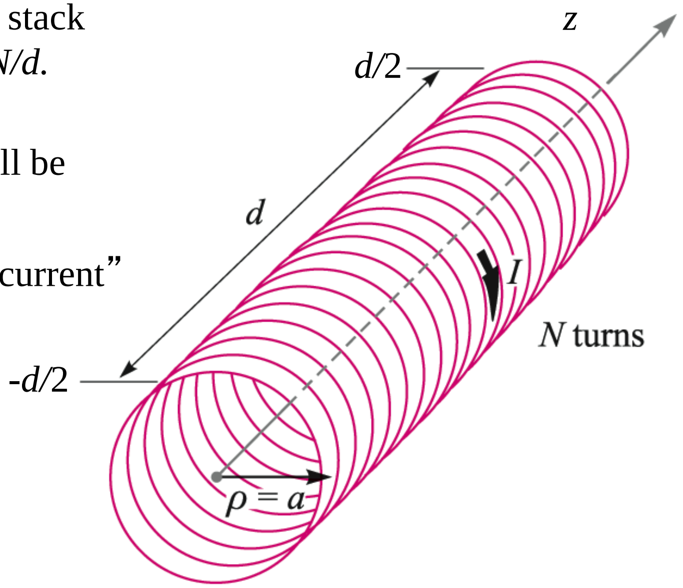
$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$



We will now use this result as a building block to construct the magnetic field on the axis of a solenoid -- formed by a stack of identical current loops, centered on the *z* axis.

On-Axis Field Within a Solenoid

We consider the single current loop field as a differential contribution to the total field from a stack of *N* closely-spaced loops, each of which carries current *I*. The length of the stack (solenoid) is *d*, so therefore the *density of turns* will be *N/d*.



Now the current in the turns within a differential length, *dz*, will be

$$\boxed{dI = \frac{N}{d} I dz}$$

We consider this as our differential "loop current"

so that the previous result for **H** from a single loop:

$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$

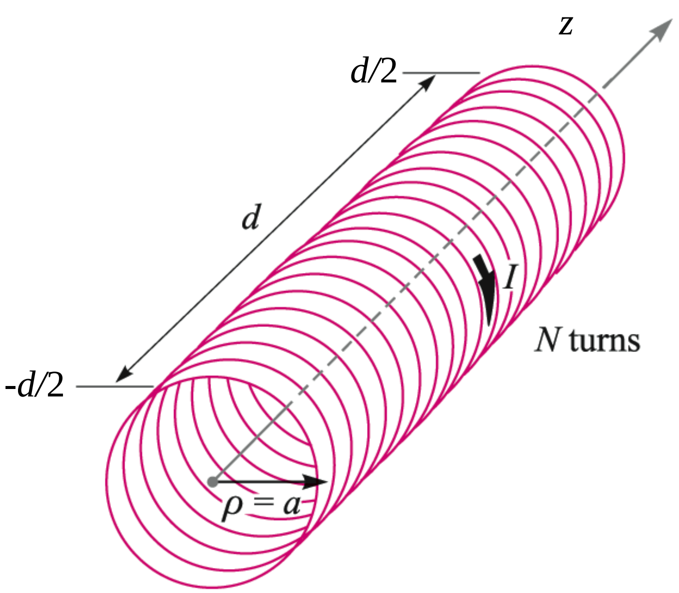
now becomes:

$$d\mathbf{H} = \frac{(N/d)Idz(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z^2)^{3/2}}$$

in which *z* is measured from the center of the coil, where we wish to evaluate the field.

Solenoid Field, Continued

The total field on the z axis at z = 0 will be the sum of the field contributions from all turns in the coil -- or the integral of $d\mathbf{H}$ over the length of the solenoid.



$$\mathbf{H} = \int d\mathbf{H} = \int_{-d/2}^{d/2} \frac{(N/d)I\,dz(\pi a^2)\mathbf{a}_z}{2\pi(a^2+z^2)^{3/2}}$$

$$= \frac{NIa^2}{2d}\mathbf{a}_z \int_{-d/2}^{d/2} \frac{dz}{(a^2+z^2)^{3/2}}$$

$$= \frac{NIa^2}{2d}\mathbf{a}_z \frac{d}{a^2\sqrt{a^2+(d/2)^2}} = \frac{NI\,\mathbf{a}_z}{2\sqrt{a^2+(d/2)^2}}$$

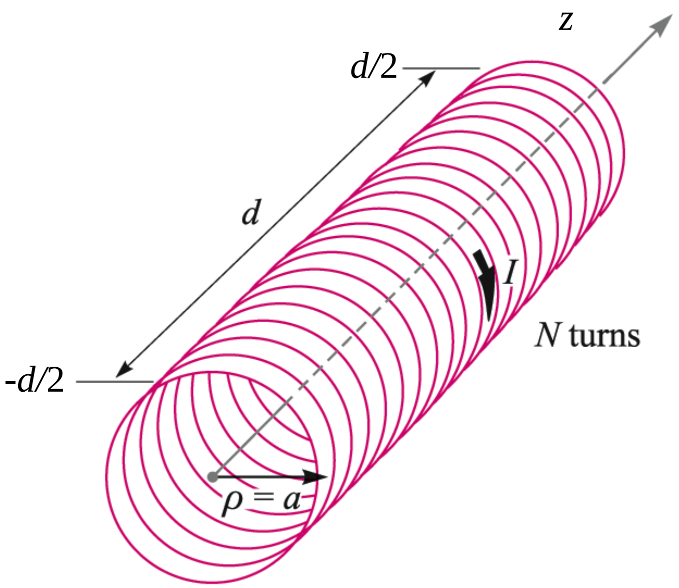
Approximation for Long Solenoids

We now have the on-axis field at the solenoid midpoint (z = 0):

$$\mathbf{H} = \frac{N I \, \mathbf{a}_z}{2\sqrt{a^2 + (d/2)^2}}$$

Note that for long solenoids, for which $d >> a$ the result simplifies to:

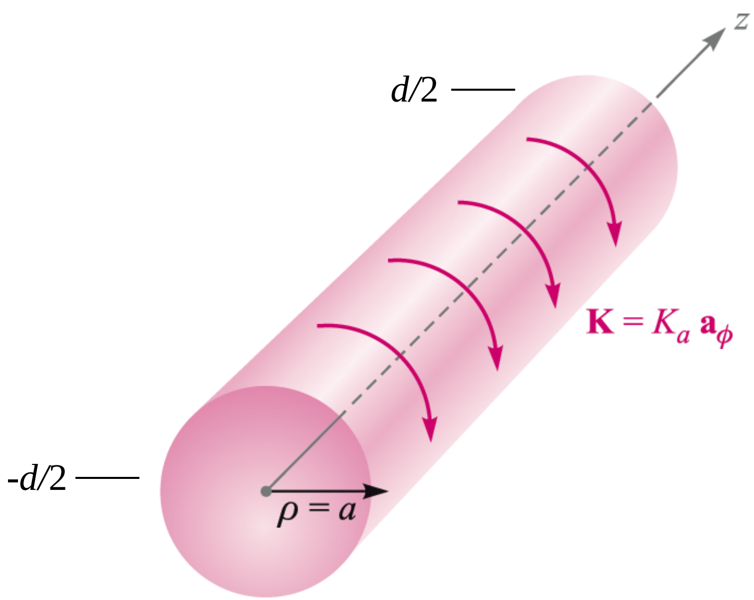
$$\boxed{\mathbf{H} \doteq \frac{N I}{d} \mathbf{a}_z \quad (d >> a)}$$



This result is valid at all on-axis positions deep within long coils -- at distances from each end of several radii.

Another Interpretation: Continuous Surface Current

The solenoid of our previous example was assumed to have many tightly-wound turns, with several existing within a differential length, *dz*. We could model such a current configuration as a continuous surface current of density $\mathbf{K} = K_a \mathbf{a}_\phi$ A/m.



$$\mathbf{K} = K_a \, \mathbf{a}_\phi = \frac{NI}{d} \mathbf{a}_\phi \quad \mathrm{A/m}$$

Therefore:

$$\mathbf{H}(\rho = z = 0) = \frac{K_a d \, \mathbf{a}_z}{2\sqrt{a^2 + (d/2)^2}}$$

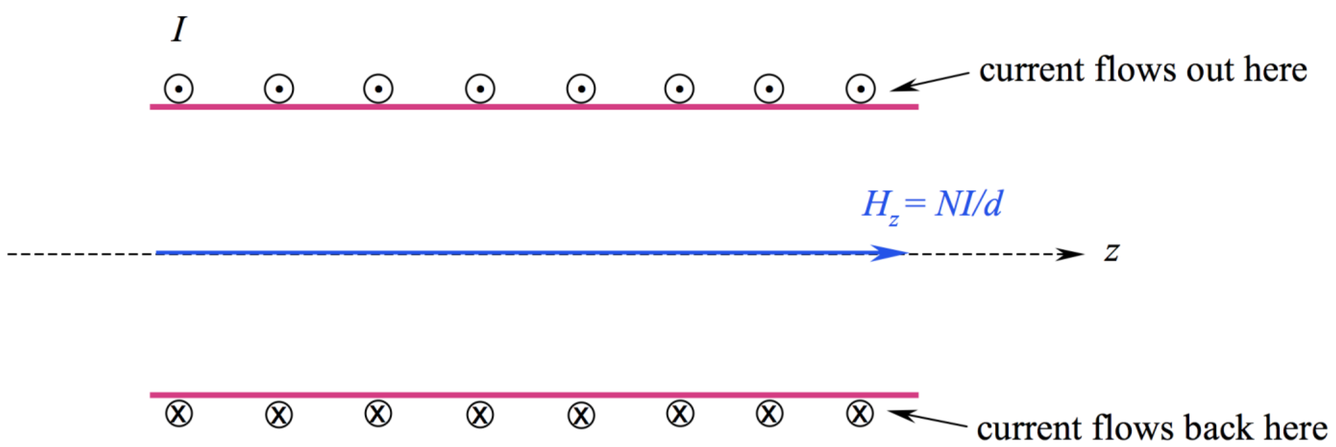
$$\doteq K_a \, \mathbf{a}_z \ (d >> a) \quad \mathrm{A/m}$$

In other words, the on-axis field magnitude near the center of a cylindrical current sheet, where current circulates around the *z* axis, and whose length is much greater than its radius, is just the surface current density.

Solenoid Field -- Off-Axis

To find the field within a solenoid, but off the z axis, we apply Ampere's Circuital Law in the following way:

The illustration below shows the solenoid cross-section, from a lengthwise cut through the z axis. Current in the windings flows in and out of the screen in the circular current path. Each turn carries current I. The magnetic field along the z axis is NI/d as we found earlier.

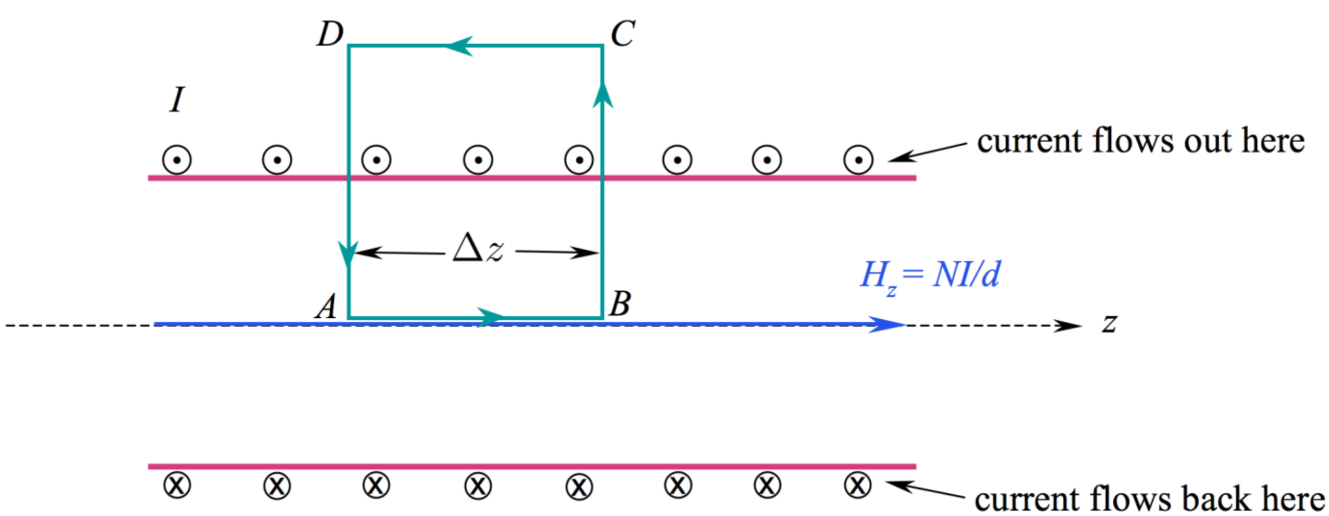


Application of Ampere's Law

Applying Ampere's Law to the rectangular path shown below leads to the following:

$$\oint \mathbf{H} \cdot d\mathbf{L} \;=\; \int_A^B H_z\, dz + \int_B^C H_\rho\, d\rho + \int_C^D H_{z,out}\, dz + \int_D^A H_\rho\, d\rho \;=\; I_{encl} \;=\; \frac{NI}{d}\Delta z$$

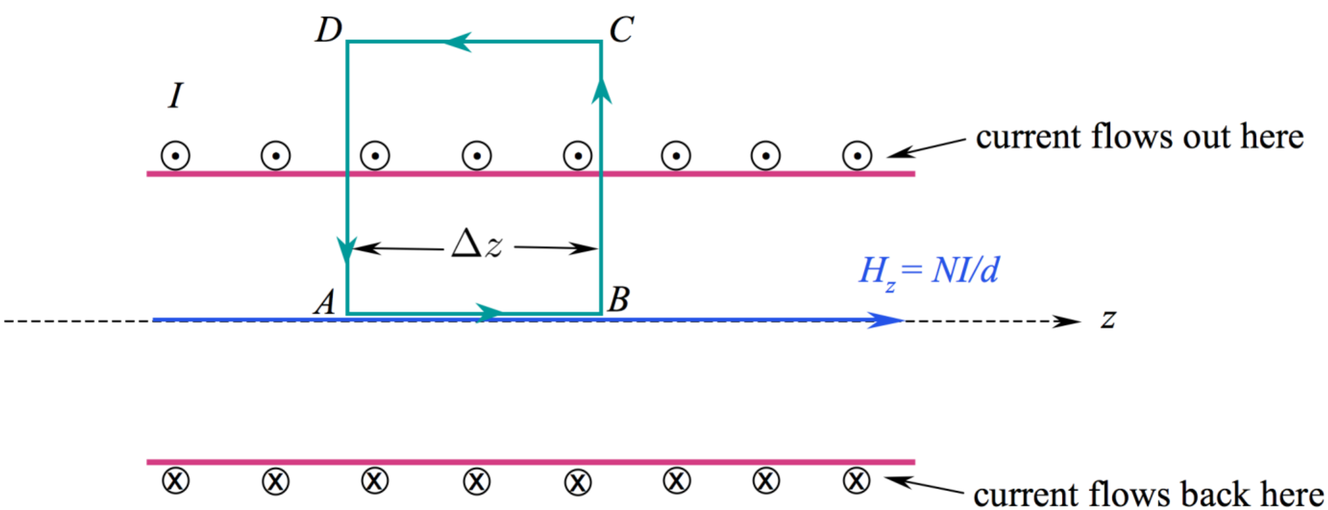
Where allowance is made for the existence of a radial H component, H_ρ



Radial Path Segments

The radial integrals will now cancel, because they are oppositely-directed, and because in the long coil, H_ρ is not expected to differ between the two radial path segments.

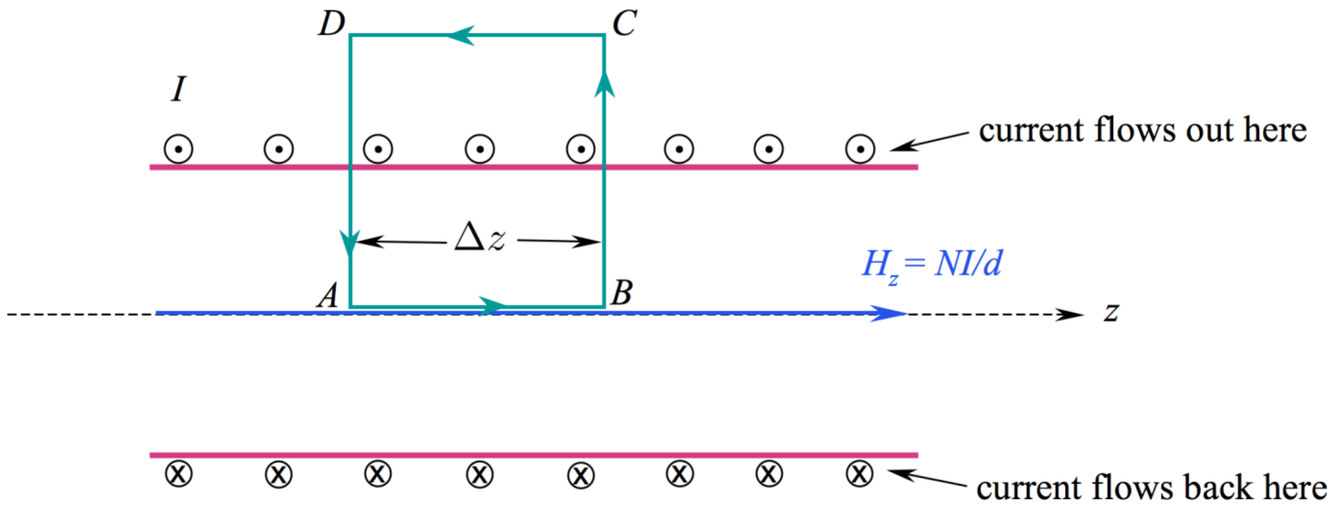
$$\oint \mathbf{H}\cdot d\mathbf{L} \;=\; \int_A^B H_z\,dz + \cancel{\int_B^C H_\rho\,d\rho} + \int_C^D H_{z,out}\,dz + \cancel{\int_D^A H_\rho\,d\rho} \;=\; I_{encl} \;=\; \frac{NI}{d}\Delta z$$



Completing the Evaluation

What is left now are the two *z* integrations, the first of which we can evaluate as shown. Since this first integral result is equal to the enclosed current, it must follow that the second integral -- and the outside magnetic field -- are zero.

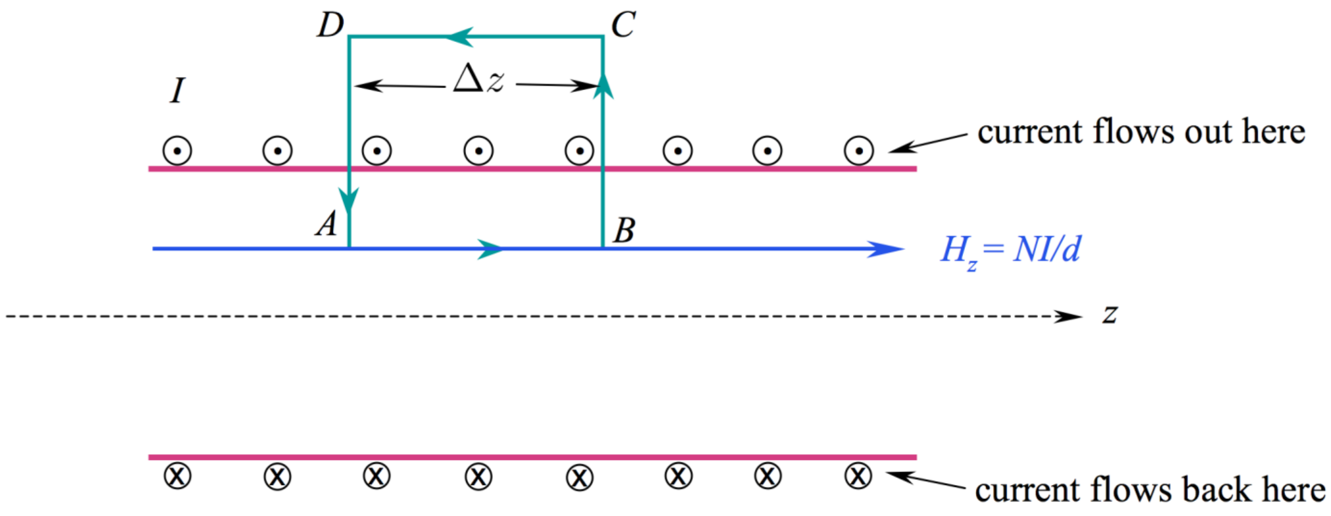
$$\oint \mathbf{H}\cdot d\mathbf{L} = \underbrace{\int_A^B H_z\,dz}_{(NI/d)\Delta z} + \cancel{\int_C^D H_{z,out}\,dz} = I_{encl} = \frac{NI}{d}\Delta z$$



Finding the Off-Axis Field

The situation does not change if the lower z-directed path is raised above the z axis. The vertical paths still cancel, and the outside field is still zero. The field along the path A to B is therefore NI/d as before.

$$\oint \mathbf{H} \cdot d\mathbf{L} \;=\; \underbrace{\int_A^B H_z\, dz}_{(NI/d)\Delta z} + \cancel{\int_C^D H_{z,out}\, dz} \;=\; I_{encl} \;=\; \frac{NI}{d}\Delta z$$

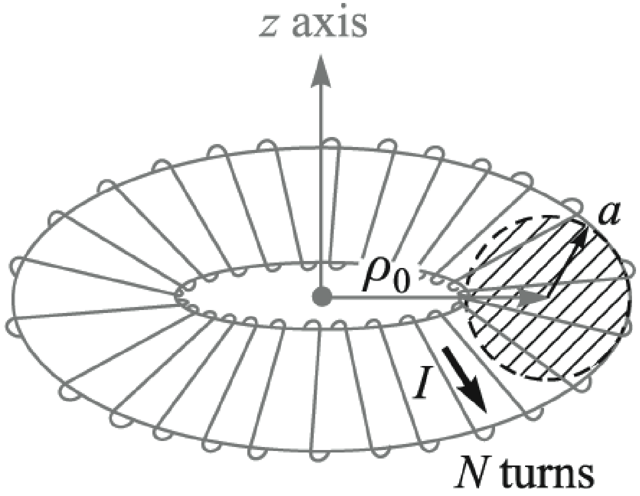


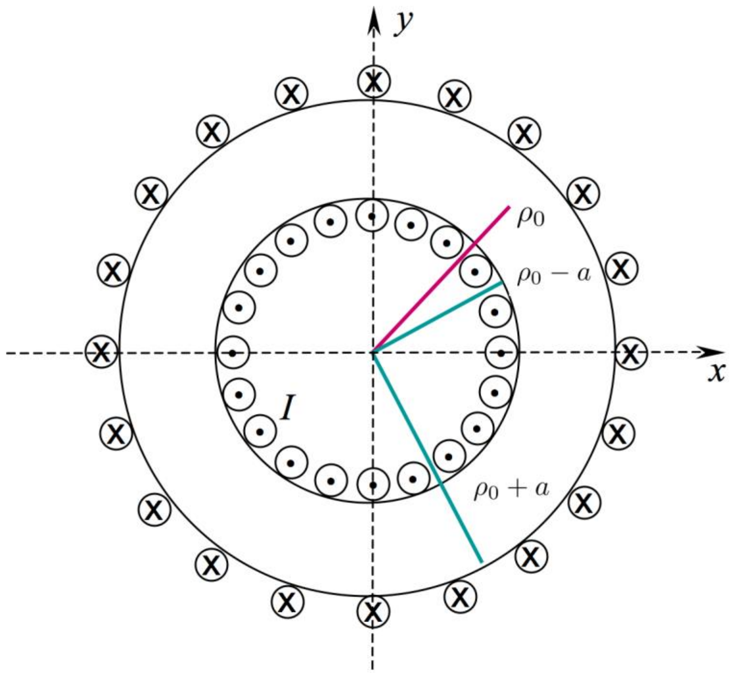
Conclusion: The magnetic field within a long solenoid is approximately constant throughout the coil cross-section, and is $H_z = NI/d$.

Toroid Magnetic Field

A toroid is a doughnut-shaped set of windings around a core material. The cross-section could be circular (as shown here, with radius *a*) or any other shape.

Below, a slice of the toroid is shown, with current emerging from the screen around the inner periphery (in the positive *z* direction). The windings are modeled as *N* individual current loops, each of which carries current *I*.



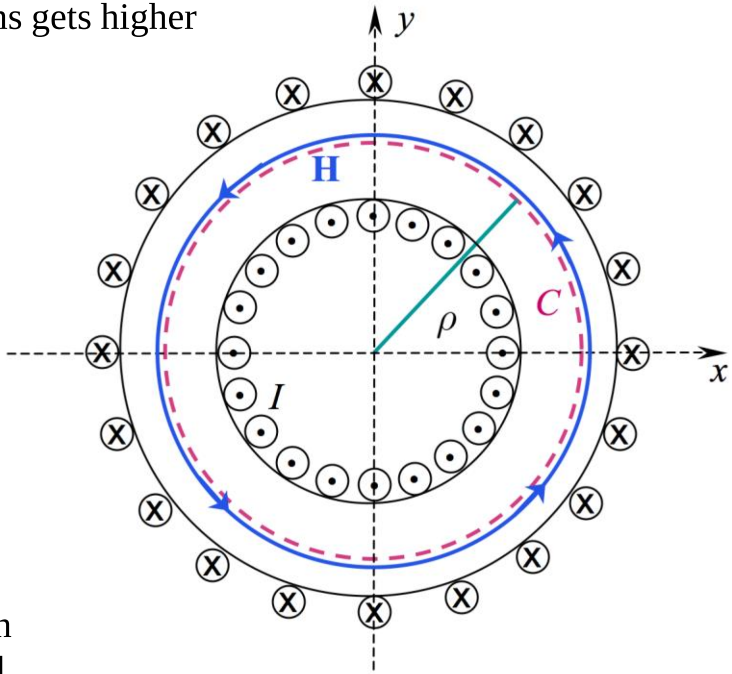


Ampere's Law as Applied to a Toroid

Ampere's Circuital Law can be applied to a toroid by taking a closed loop integral around the circular contour *C* at radius ρ. Magnetic field **H** is presumed to be circular, and a function of radius only at locations within the toroid that are not too close to the individual windings. Under this condition, we would assume:

$$\mathbf{H} = H_\phi \, \mathbf{a}_\phi$$

This approximation improves as the density of turns gets higher (using more turns with finer wire).



Ampere's Law now takes the form:

$$\oint_C \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho\, H_\phi = I_{encl} = NI$$

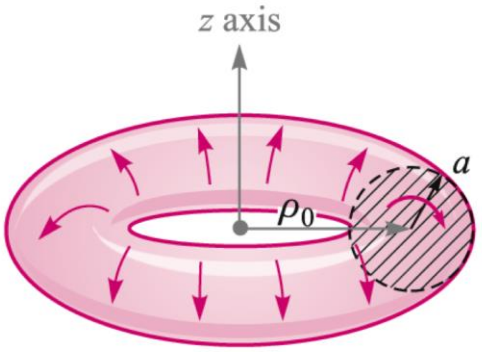
so that….

$$H_\phi = \frac{NI}{2\pi\rho} \qquad (\rho_0 - a < \rho < \rho_0 + a)$$

Performing the same integrals over contours drawn in the regions $\rho < \rho_0 - a$ or $\rho > \rho_0 + a$ will lead to *zero* magnetic field there, because no current is enclosed in either case.

Surface Current Model of a Toroid

Consider a sheet current molded into a doughnut shape, as shown. The current density at radius $\rho_0 - a$ crosses the *xy* plane in the *z* direction and is given in magnitude by K_a



Ampere's Law applied to a circular contour *C* inside the toroid (as in the previous example) will take the form:

$$\oint_C \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = I_{encl} = 2\pi(\rho_0 - a)K_a$$

leading to…

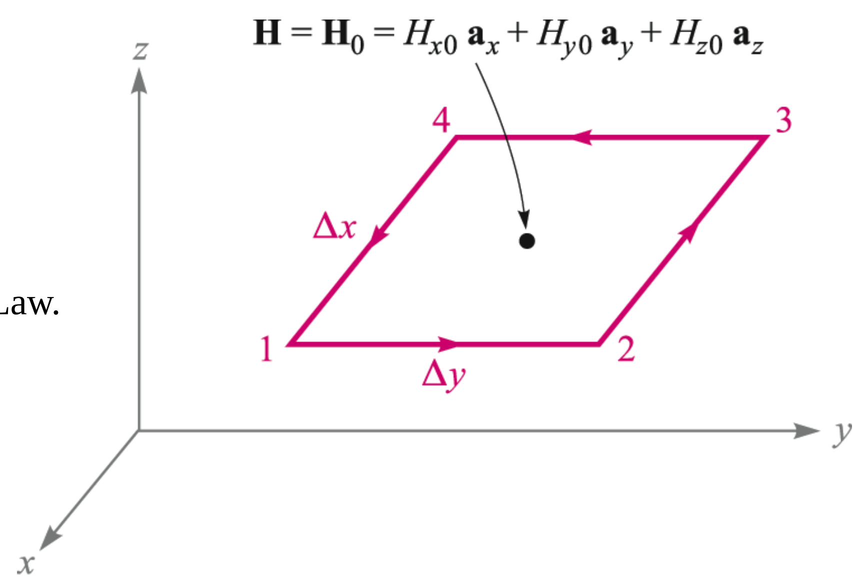
$$\boxed{H_\phi = \frac{\rho_0 - a}{\rho} K_a}$$

inside the toroid…. and the field is zero outside as before.

Ampere's Law as Applied to a Small Closed Loop.

Consider magnetic field **H** evaluated at the point shown in the figure. We can approximate the field over the closed path 1234 by making appropriate adjustments in the value of **H** along each segment.

The objective is to take the closed path integral and ultimately obtain the point form of Ampere's Law.



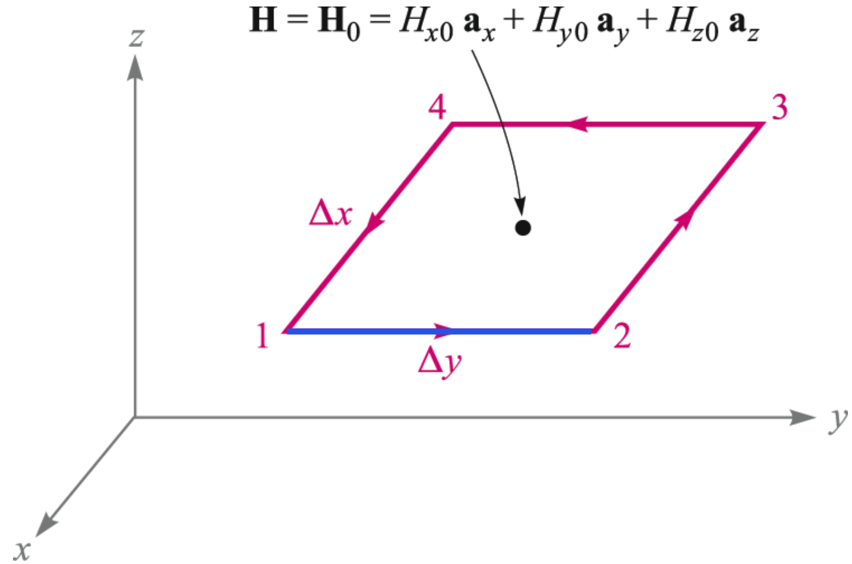
Approximation of **H** Along One Segment

Along path 1-2, we may write:

$$(\mathbf{H} \cdot \Delta \mathbf{L})_{1-2} = H_{y,1-2} \Delta y$$

where:

$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)$$



And therefore:

$$(\mathbf{H} \cdot \Delta \mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

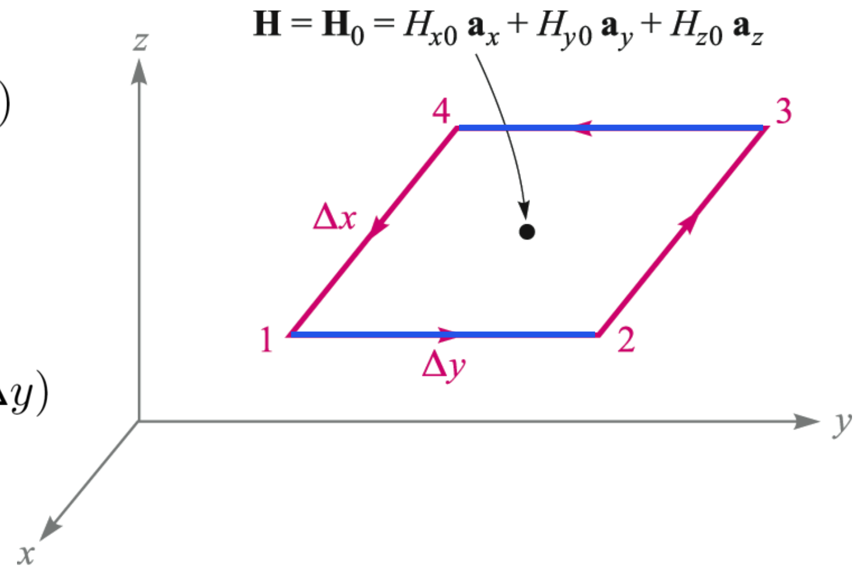
Contributions of *y*-Directed Path Segments

The contributions from the front and back sides will be:

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)(\Delta y)$$

The contribution from the opposite side is:

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{3-4} \doteq \left(H_{y0} - \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)(-\Delta y)$$



Note the path directions as specified in the figure, and how these determine the signs used .

This leaves the left and right sides…..

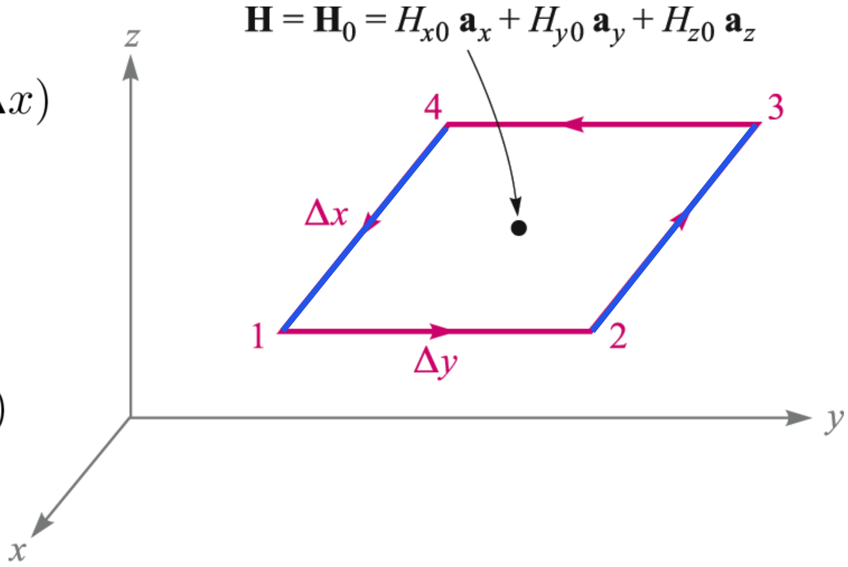
Contributions of *x*-Directed Path Segments

Along the right side (path 2-3):

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{2-3} \doteq \left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)(-\Delta x)$$

…and the contribution from the left side (path 4-1) is:

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{4-1} \doteq \left(H_{x0} - \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)(\Delta x)$$



The next step is to add the contributions of all four sides to find the closed path integral:

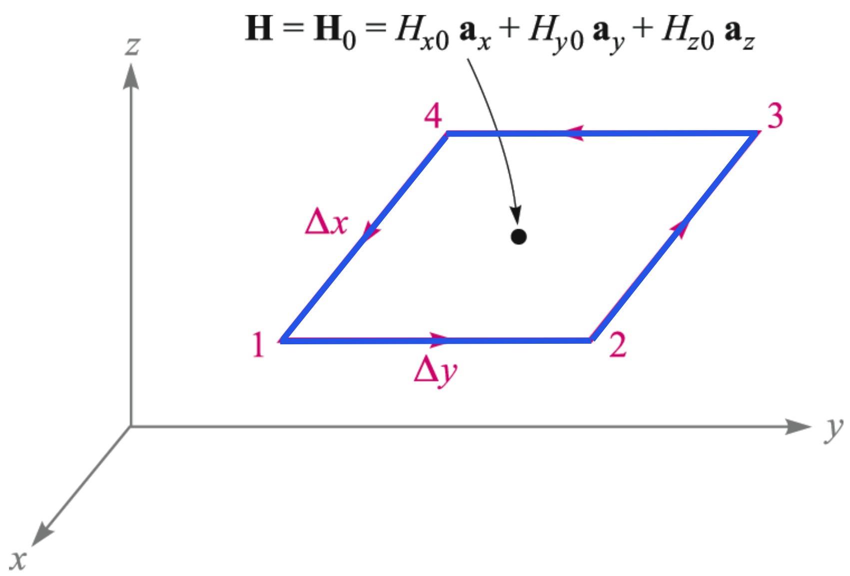
Net Closed Path Integral

The total integral will now be the sum:

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq (\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} + (\mathbf{H} \cdot \Delta\mathbf{L})_{2-3} + (\mathbf{H} \cdot \Delta\mathbf{L})_{3-4} + (\mathbf{H} \cdot \Delta\mathbf{L})_{4-1}$$

and using our previous results, the becomes:

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x \Delta y$$



Relation to the Current Density

By Ampere's Law, the closed path integral of **H** is equal to the enclosed current, approximated in this case by the current density at the center, multiplied by the loop area:

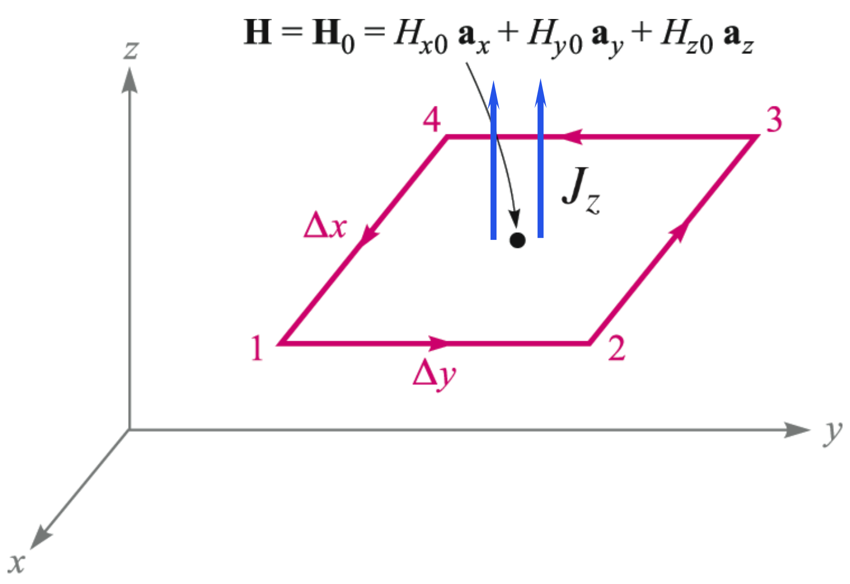
$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y \doteq J_z \Delta x \Delta y$$

Dividing by the loop area, we now have:

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

The expression becomes exact as the loop area approaches zero:

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$



Other Loop Orientations

The same exercise can be carried with the rectangular loop in the other two orthogonal orientations. The results are:

Loop in *yz* plane:
$$\lim_{\Delta y,\Delta z\to 0}\frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta y\Delta z}=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=J_x$$

Loop in *xz* plane:
$$\lim_{\Delta z,\Delta x\to 0}\frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta z\Delta x}=\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=J_y$$

Loop in *xy* plane:
$$\lim_{\Delta x,\Delta y\to 0}\frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta x\Delta y}=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=J_z$$

This gives all three components of the current density field.

Curl of a Vector Field

The previous exercise resulted in the rectangular coordinate representation of the *Curl* of **H**.

In general, the curl of a vector field is another field that is normal to the original field.

The curl component in the direction *N*, normal to the plane of the integration loop is:

$$\boxed{(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N}}$$

where ΔS_N is the planar area enclosed by the closed line integral.

The direction of *N* is taken using the right-hand convention: With fingers of the right hand oriented in the direction of the path integral, the thumb points in the direction of the normal (or curl).

Curl in Rectangular Coordinates

Assembling the results of the rectangular loop integration exercise, we find the vector field that comprises curl **H**:

$$\text{curl}\,\mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

An easy way to calculate this is to evaluate the following determinant:

$$\text{curl}\,\mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

which we see is equivalent to the cross product of the del operator with the field:

$$\text{curl}\,\mathbf{H} = \nabla \times \mathbf{H}$$

Curl in Other Coordinate Systems

…a little more complicated!

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)}$$

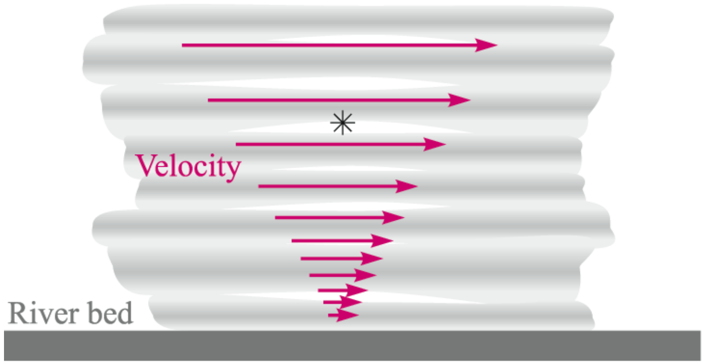
$$\nabla \times \mathbf{H} = \frac{1}{r \sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)}$$

Look these up as needed….

Visualization of Curl

Consider placing a small “paddle wheel” in a flowing stream of water, as shown below. The wheel axis points into the screen, and the water velocity decreases with increasing depth.

The wheel will rotate clockwise, and give a curl component that points into the screen (right-hand rule).



Positioning the wheel at all three orthogonal orientations will yield measurements of all three components of the curl. Note that the curl is directed normal to both the field and the direction of its variation.

Another Maxwell Equation

It has just been demonstrated that:

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J}$$

…..which is in fact one of Maxwell's equations for static fields:

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

This is Ampere's Circuital Law in point form.

….and Another Maxwell Equation

We already know that for a *static* electric field:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

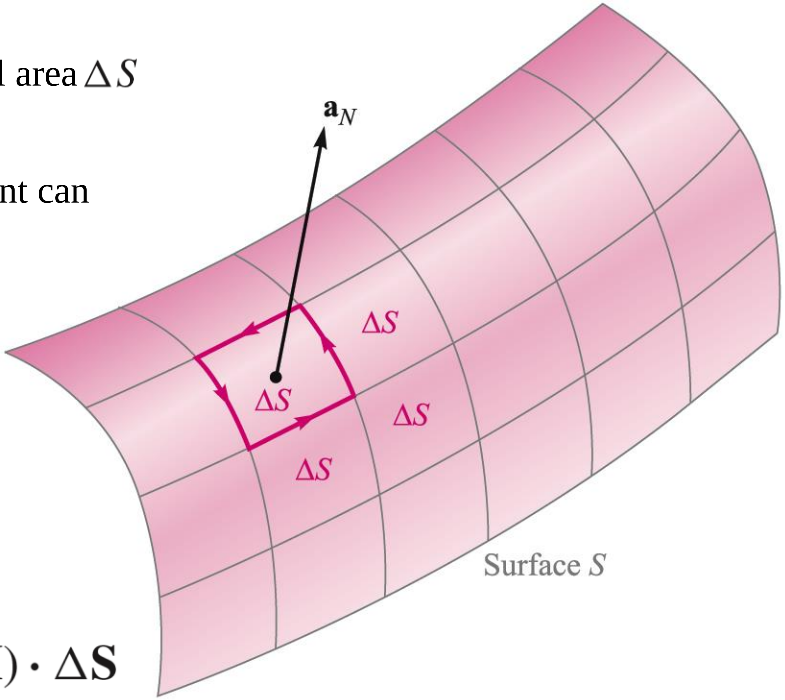
This means that: $\boxed{\nabla \times \mathbf{E} = 0}$ (applies to a static electric field)

Recall the condition for a conservative field: that is, its closed path integral is zero everywhere.

Therefore, a field is conservative if it has *zero curl* at all points over which the field is defined.

Curl Applied to Partitions of a Large Surface

Surface S is paritioned into sub-regions, each of small area ΔS



The curl component that is normal to a surface element can be written using the definition of curl:

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

or:

$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S}$$

We now apply this to every partition on the surface, and add the results….

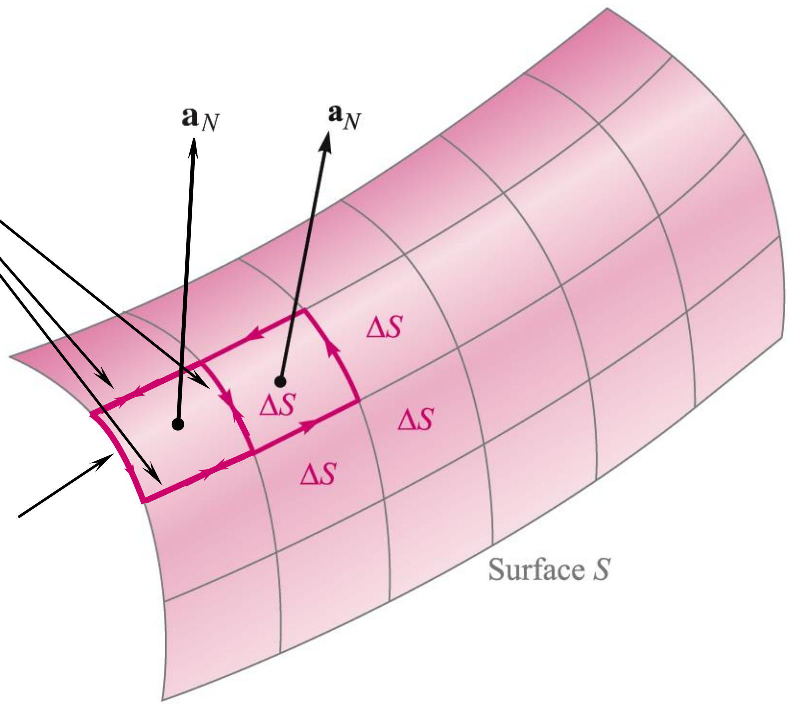
Adding the Contributions

Cancellation here:

We now evaluate and add the curl contributions from all surface elements, and note that adjacent path integrals will all cancel!

This means that the only contribution to the overall path integral will be around the outer periphery of surface *S*.

No cancellation here:



Using our previous result, we now write:

$$\sum_{\substack{\text{all surface}\\ \text{elements}}} \oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq \sum_{\substack{\text{all surface}\\ \text{elements}}} \nabla \times \mathbf{H} \cdot \mathbf{a}_N \Delta S$$

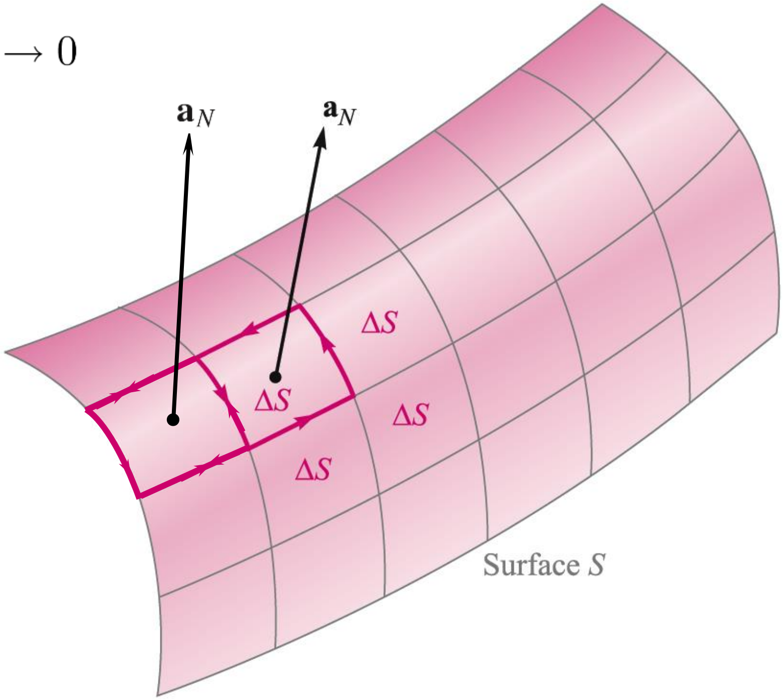
Stokes' Theorem

We now take our previous result, and take the limit as $\Delta S \rightarrow 0$

$$\underbrace{\sum_{\substack{\text{all surface}\\ \text{elements}}} \oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}_{} \doteq \underbrace{\sum_{\substack{\text{all surface}\\ \text{elements}}} \nabla \times \mathbf{H} \cdot \mathbf{a}_N \Delta S}_{}$$

In the limit, this side becomes the path integral of **H** over the outer perimeter because all interior paths cancel

In the limit, this side becomes the integral of the curl of **H** over surface *S*



The result is Stokes' Theorem

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}}$$

This is a valuable tool to have at our disposal, because it gives us two ways to evaluate the same thing!

Obtaining Ampere's Circuital Law in Integral Form, using Stokes' Theorem

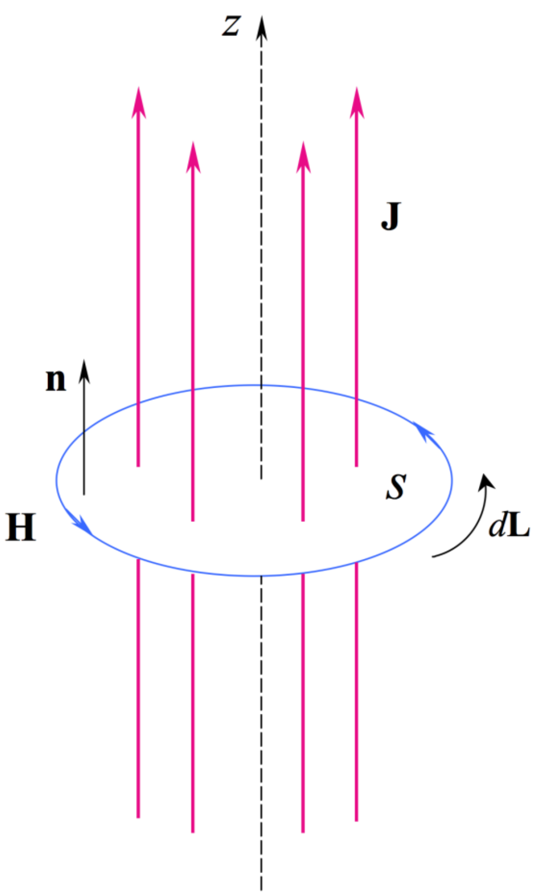
Begin with the point form of Ampere's Law for static fields:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Integrate both sides over surface *S*:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$

..in which the far right hand side is found from the left hand side using Stokes' Theorem. The closed path integral is taken around the perimeter of *S*. Again, note that we use the right-hand convention in choosing the direction of the path integral.



The center expression is just the net current through surface *S*, so we are left with the integral form of Ampere's Law:

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

Magnetic Flux and Flux Density

We are already familiar with the concept of electric flux:

$$\Psi = \int_s \mathbf{D} \cdot d\mathbf{S} \quad \text{Coulombs}$$

in which the electric flux density in free space is: $\mathbf{D} = \epsilon_0 \mathbf{E} \quad \mathrm{C/m^2}$

and where the free space permittivity is $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

In a similar way, we can define the *magnetic flux* in units of Webers (Wb):

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{S} \quad \text{Webers}$$

in which the *magnetic flux density* (or *magnetic* induction) in free space is: $\mathbf{B} = \mu_0 \mathbf{H} \quad \mathrm{Wb/m^2}$

and where the free space *permeability* is $\mu_0 = 4\pi \times 10^{-7}$ H/m

This is a *defined* quantity, having to do with the definition of the ampere (we will explore this later).

A Key Property of **B**

If the flux is evaluated through a closed surface, we have in the case of electric flux, Gauss' Law:

$$\Psi_{net} = \oint_s \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

If the same were to be done with magnetic flux density, we would find:

$$\Phi_{net} = \oint_s \mathbf{B} \cdot d\mathbf{S} = 0$$

The implication is that (for our purposes) there are no magnetic charges -- specifically, *no point sources of magnetic field exist*. A hint of this has already been observed, in that magnetic field lines always close on themselves.

Another Maxwell Equation

We may rewrite the closed surface integral of **B** using the divergence theorem, in which the right hand integral is taken over the volume surrounded by the closed surface:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \; dv = 0$$

Because the result is zero, it follows that

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

This result is known as Gauss' Law for the magnetic field in point form.

Maxwell's Equations for Static Fields

We have now completed the derivation of Maxwell's equations for no time variation. In point form, these are:

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_\nu \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}}$$

Gauss' Law for the electric field

Conservative property of the static electric field

Ampere's Circuital Law

Gauss' Law for the Magnetic Field

where, in free space:

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}}$$

$$\boxed{\mathbf{B} = \mu_0 \mathbf{H}}$$

Significant changes in the above four equations will occur when the fields are allowed to vary with time, as we'll see later.

Maxwell's Equations in Large Scale Form

The divergence theorem and Stokes' theorem can be applied to the previous four point form equations to yield the integral form of Maxwell's equations for static fields:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_\nu d\nu$$

Gauss' Law for the electric field

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

Conservative property of the static electric field

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

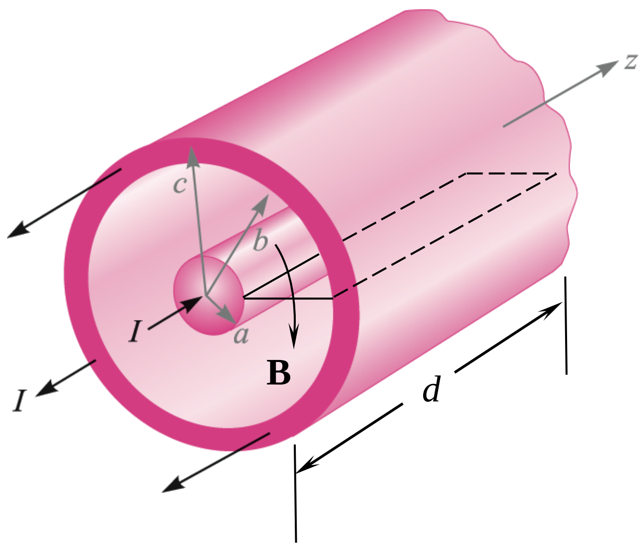
Ampere's Circuital Law

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Gauss' Law for the magnetic field

Example: Magnetic Flux Within a Coaxial Line

Consider a length d of coax, as shown here. The magnetic field strength between conductors is:



$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

and so: $$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

The magnetic flux is now the integral of **B** over the flat surface between radii a and b, and of length d along z:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi$$

The result is: $$\Phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a}$$

The coax line thus "stores" this amount of magnetic flux in the region between conductors. This will have importance when we discuss inductance in a later lecture.

Scalar Magnetic Potential

We are already familiar with the relation between the scalar electric potential and electric field:

$$\mathbf{E} = -\nabla V$$

So it is tempting to define a scalar magnetic potential such that:

$$\mathbf{H} = -\nabla V_m$$

This rule must be consistent with Maxwell's equations, so therefore:

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$$

But the curl of the gradient of any function is identically zero! Therefore, the scalar magnetic potential is valid only in regions where the current density is zero (such as in free space).

So we define scalar magnetic potential with a condition:

$$\boxed{\mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)}$$

Further Requirements on the Scalar Magnetic Potential

The other Maxwell equation involving magnetic field must also be satisfied. This is:

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0 \qquad \text{in free space}$$

Therefore: $$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

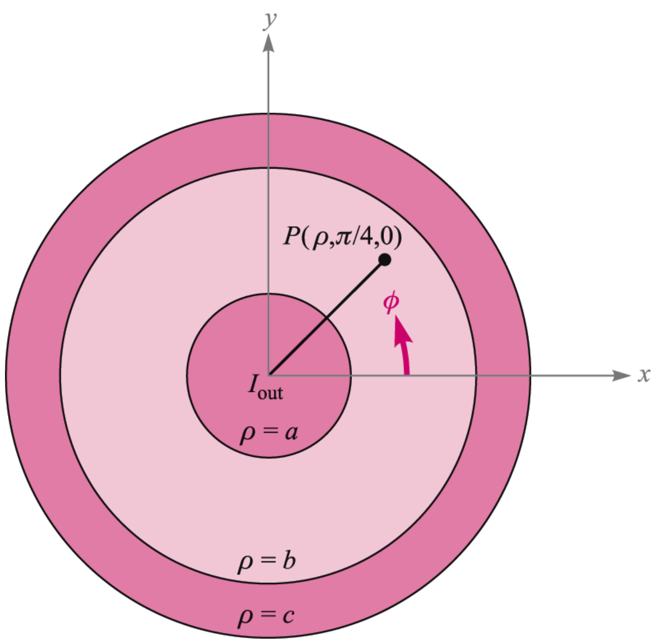
..and so the scalar magnetic potential satisfies Laplace's equation (again with the restriction that current density must be zero:

$$\boxed{\nabla^2 V_m = 0 \quad (\mathbf{J} = 0)}$$

Example: Coaxial Transmission Line

With the center conductor current flowing out of the screen, we have

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$



Thus: $$\frac{I}{2\pi\rho} = -\nabla V_m \Big|_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$

So we solve: $$\frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

.. and obtain: $$\underline{V_m = -\frac{I}{2\pi}\phi}$$

where the integration constant has been set to zero

Ambiguities in the Scalar Potential

The scalar potential is now:

$$V_m = -\frac{I}{2\pi}\phi$$

where the potential is zero at $\phi = 0$

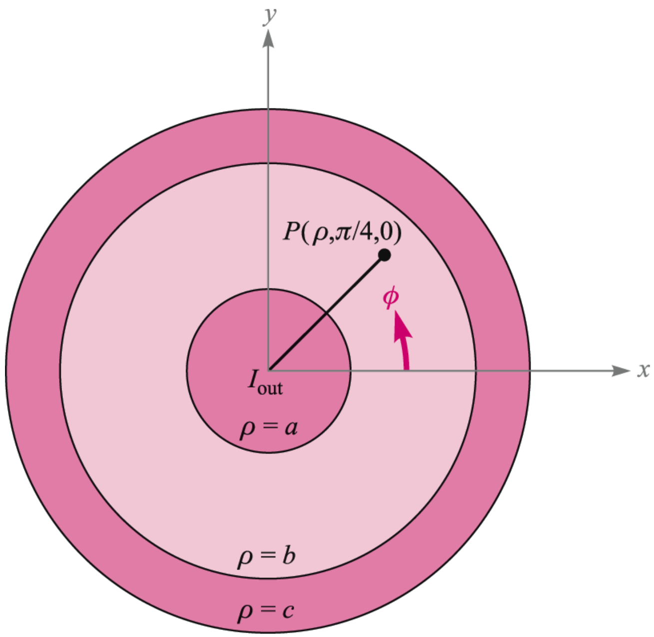
At point P ($\phi = \pi/4$) the potential is

$$V_{mP}(\phi = \pi/4) = -I/8$$

But wait! As ϕ increases to $\phi = 2\pi$
we have returned to the same physical location, and
the potential has a new value of -*I*.

In general, the potential at *P* will be multivalued, and will acquire a new value after each full rotation in the *xy* plane:

$$V_{mP} = \frac{I}{2\pi}\left(2n - \tfrac{1}{4}\right)\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$



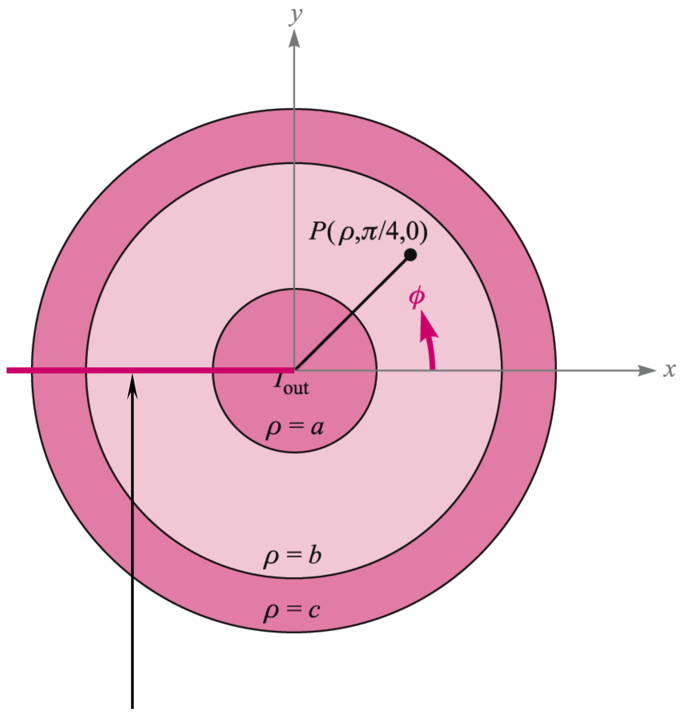
Overcoming the Ambiguity

To remove the ambiguity, we construct a mathematical barrier at any value of phi. The angle domain cannot cross this barrier in either direction, and so the potential function is restricted to angles on either side. In the present case we choose the barrier to lie at $\phi = \pi$ so that

$$V_m = -\frac{I}{2\pi}\phi \quad (-\pi < \phi < \pi)$$

The potential at point *P* is now single-valued:

$$V_{mP} = -\frac{I}{8} \quad \left(\phi = \frac{\pi}{4}\right)$$



Barrier at $\phi = \pi$

Vector Magnetic Potential

We make use of the Maxwell equation: $\nabla \cdot \mathbf{B} = 0$

.. and the fact that the divergence of the curl of any vector field is identically zero (show this!)

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

This leads to the definition of the *magnetic vector potential*, **A**:

$$\underline{\mathbf{B} = \nabla \times \mathbf{A}}$$

Thus: $\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A}$

and Ampere's Law becomes $\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$

Equation for the Vector Potential

We start with: $$\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$$

Then, introduce a vector identity that defines the *vector Laplacian*:

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Using a (lengthy) procedure (see Sec. 7.7) it can be proven that $\underline{\nabla \cdot \mathbf{A} = 0}$

We are therefore left with

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$$

The Direction of **A**

We now have $$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$$

In rectangular coordinates:

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

(not so simple in the other coordinate systems)

The equation separates to give: $$\nabla^2 A_x = -\mu_0 J_x$$

$$\nabla^2 A_y = -\mu_0 J_y$$

$$\nabla^2 A_z = -\mu_0 J_z$$

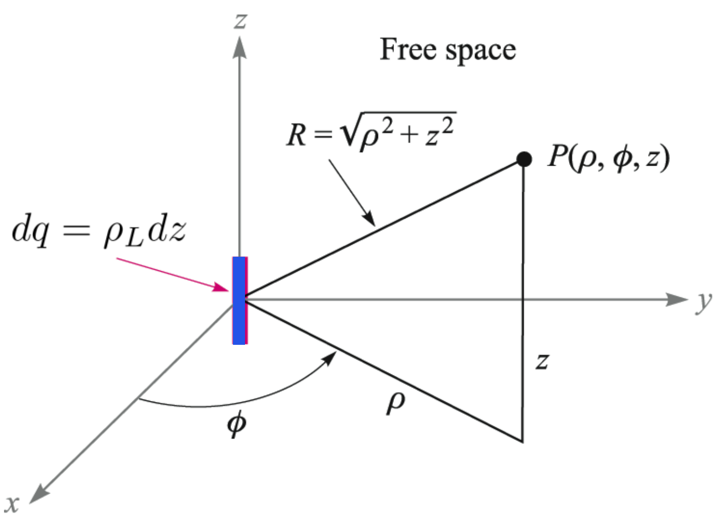
This indicates that the direction of **A** will be the same as that of the current to which it is associated.

The vector field, **A**, existing in all space, is sometimes described as being a "fuzzy image" of its generating current.

Expressions for Potential

Consider a differential elements, shown here. On the left is a point charge represented by a differential length of line charge. On the right is a differential current element. The setups for obtaining potential are identical between the two cases.

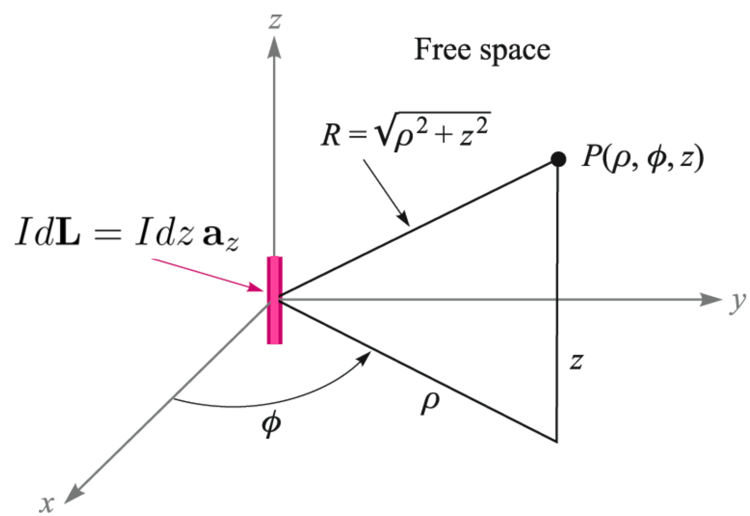
Line Charge



Scalar Electrostatic Potential

$$\boxed{dV = \frac{dq}{4\pi\epsilon_0 R} = \frac{\rho_L dL}{4\pi\epsilon_0 R}}$$

Line Current



Vector Magnetic Potential

$$\boxed{d\mathbf{A} = \frac{\mu_0 Id\mathbf{L}}{4\pi R} = \frac{\mu_0 Idz\,\mathbf{a}_z}{4\pi R}}$$

General Expressions for Vector Potential

For large scale charge or current distributions, we would sum the differential contributions by integrating over the charge or current, thus:

$$V = \int \frac{\rho_L \, dL}{4\pi\epsilon_0 R}$$

and

$$\boxed{\mathbf{A} = \oint \frac{\mu_0 I \, d\mathbf{L}}{4\pi R}}$$

The closed path integral is taken because the current must close on itself to form a complete circuit.

For surface or volume current distributions, we would have, respectively:

$$\boxed{\mathbf{A} = \int_S \frac{\mu_0 \mathbf{K} \, dS}{4\pi R}}$$

or

$$\boxed{\mathbf{A} = \int_{\mathrm{vol}} \frac{\mu_0 \mathbf{J} \, dv}{4\pi R}}$$

in the same manner that we used for scalar electric potential.

Example

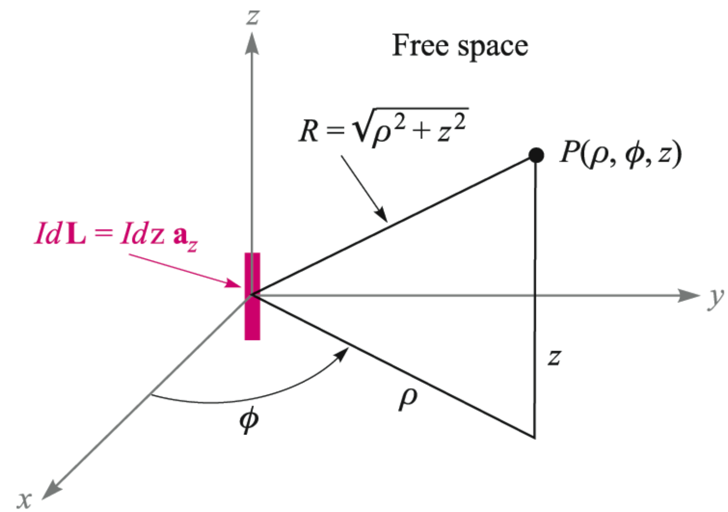
We continue with the differential current element as shown here:

In this case

$$dA = \frac{\mu_0 I \, d\mathbf{L}}{4\pi R}$$

becomes at point *P*:

$$d\mathbf{A} = \frac{\mu_0 I \, dz \, \mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$



Now, the curl is taken in cylindrical coordinates:

$$d\mathbf{H} = \frac{1}{\mu_0}\nabla \times d\mathbf{A} = \frac{1}{\mu_0}\left(-\frac{\partial dA_z}{\partial \rho}\right)\mathbf{a}_\phi = \frac{I\,dz}{4\pi}\frac{\rho}{(\rho^2 + z^2)^{3/2}}\mathbf{a}_\phi$$

This is the same result as found using the Biot-Savart Law (as it should be)